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Exact soliton solutions for $(2+1)-$dimensional dispersive long wave equation by using the first integral method

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ABSTRACT

In this paper, the first integral method is used to construct exact traveling wave solutions of $(2+1)-$ dimensional dispersive long wave equation. The first integral method is an efficient method for obtaining exact solutions some of nonlinear partial differential equations. This method can be applied to nonintegrable equations as well as to integrable ones.

Keywords: First integral method; $(2+1)-$dimensional dispersive long wave equation

1. Introduction

Nonlinear evolution equations (NLEEs) have been the subject of study in various branches of mathematical-physical sciences such as physics, biology, chemistry. The analytical solutions of such equations are of fundamental importance since a lot of mathematical physical models are described by NLEEs. Among the possible solutions to NLEEs, certain special form solutions may depend only on a single combination of variables such as traveling wave variables. In the literature, there is a wide variety of approaches to nonlinear problems for constructing traveling wave solutions, such as tanh-sech method $[1,2,3]$, extended tanh method $[4,5,6]$, hyperbolic function method $[7]$, sine-cosine method $[8,9,10]$, Jacobi elliptic function expansion method $[11]$, and the first integral method $[12,13]$.

The first integral method was first proposed by Feng $[12]$ in solving Burgers-KdV equation which is based on the ring theory of commutative algebra. Recently, this useful method is widely used by many such as in $[14,15]$ and by the reference therein. The present paper investigates for the first time the applicability and effectiveness of the first integral method on high-dimensional partial differential system.

2. The first integral method(FIM)

Consider the nonlinear partial differential equation:

$$F(u,u_x,u_y,u_t,u_{xx},u_{xy},u_{xt},...)=0, \tag{2.1}$$

where $u=u(x,y,t)$ is the solution of (1). We use the traveling wave transformations,

$$u(x,y,t)=f(\xi), \tag{2.2}$$

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where $\xi = kx + ly - \lambda t$. Based on this we obtain

$$\frac{\partial}{\partial t}(.) = -\lambda \frac{\partial}{\partial \xi}(.), \quad \frac{\partial}{\partial x}(.) = k\frac{\partial}{\partial \xi}(.), \quad \frac{\partial}{\partial y}(.) = l\frac{\partial}{\partial \xi}(.), \quad \frac{\partial^2}{\partial x^2}(.) = k^2\frac{\partial^2}{\partial \xi^2}(.), \quad .... \tag{2.3}$$

We use (3) to change the nonlinear partial differential equation (1) to nonlinear ordinary differential equation

$$G(f(\xi), \frac{\partial f(\xi)}{\partial \xi}, \frac{\partial^2 f(\xi)}{\partial \xi^2}, ...) = 0. \tag{2.4}$$

Next,we introduce a new independent variable

$$X(\xi) = f(\xi), \qquad Y(\xi) = \frac{\partial f(\xi)}{\partial \xi} \tag{2.5}$$

which leads a system of nonlinear ordinary differential equations

$$\frac{\partial X(\xi)}{\partial \xi} = Y(\xi), \tag{2.6}$$

$$\frac{\partial Y(\xi)}{\partial \xi} = F_1(X(\xi), Y(\xi)).$$

By the qualitative theory of ordinary differential equations [13] , if we can find the integrals to (6) under the same conditions, then the general solutions to (6) can be solved directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way for telling us what these first integrals are. We will apply the Hilbert-Nullstellensatz theorem to obtain one first integral to (6), which reduces (4) to a first-order integrable ordinary differential equation. An exact solution to (1) is then obtained by solving this equation. Now, let us recall the Division Theorem:

Theorem 2.1 (Division Theorem)**.** *Suppose that $P(w,z)$ and $Q(w,z)$ are polynomials in $C[w,z]$; and $P(w,z)$ is irreducible in $C[w,z]$; If $Q(w,z)$ vanishes at all zero points of $P(w,z)$, then there exists a polynomial $G(w,z)$ in $C[w,z]$ such that*

$$Q(w,z) = P(w,z)G(w,z)$$

The Division theorem follows immediately from the Hilbert-Nullstellensatz theorem [16]:

Theorem 2.2 (Hilbert-Nullstellensatz Theorem)**.** *Let k be a field and L an algebraic closure of k.*
(i) Every ideal γ of $k[X_1, ..., X_n]$ not containing 1 admits at least one zero in L^n.
(ii) Let $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$ be two elements of L^n; for the set of polynomials of $k[X_1, ..., X_n]$ zero at x to be identical with the set of polynomials of $k[X_1, ..., X_n]$ zero at y, it in necessary and sufficient that there exists a k-automorphism s of L such that $y_i = s(x_i)$ for $1 \le i \le n$.
(iii) For an ideal α of $k[X_1, ..., X_n]$ to be maximal, it is necessary and sufficient that there exists and x in L^n such that α is the set of polynomials of $k[X_1, ..., X_n]$ zero at x.
(iv) For a polynomial Q of $k[X_1, ..., X_n]$ to be zero on the set of zeros in L^n of an ideal γ of $k[X_1, ..., X_n]$, it is necessary and sufficient that there exist and integer $m > 0$ such that $Q^m \in \gamma$.

3. $(2 + 1)-$dimensional dispersive long wave equation

Let us apply the first integral method to find out exact solutions of the $(2 + 1)-$dimensional dispersive long wave equation [17] with source

$$u_{yt} + v_{xx} + u_x u_y + u u_{xy} = 0, \tag{3.1}$$

$$v_t + u_x + (uv)_x + u_{xxy} = 0,$$

where $u = u(x, y, t)$, $v = v(x, y, t)$ are unknown functions.
We use the traveling wave transformations

$$u(x, y, t) = u(\xi), \;\; v(x, y, t) = v(\xi), \;\; \xi = kx + ly - \lambda t. \tag{3.2}$$

Using (8) system (7) becomes

$$-\lambda l u'' + k^2 v'' + kl(u')^2 + kluu'' = 0, \tag{3.3}$$
$$-\lambda v' + ku' + k(uv)' + k^2 l u''' = 0. \tag{3.4}$$

Integrating (9) twice with respect to ξ, then we have

$$-\lambda l u + k^2 v + \frac{kl}{2} u^2 = R_1, \tag{3.5}$$

where R_1 is second integration constant and the first one is taken to zero. Rewrite this equation as follows

$$v(\xi) = \frac{R_1}{k^2} + \frac{l\lambda}{k^2} u(\xi) - \frac{l}{2k} u^2(\xi). \tag{3.6}$$

Inserting Eq. (12) into Eq. (10) yields

$$(k + \frac{R_1}{k} - \frac{l\lambda^2}{k^2})u' + \frac{(\lambda l + 2kl)}{k} uu' - \frac{3l}{2} u^2 u' + k^2 l u''' = 0. \tag{3.7}$$

Integrating Eq. (13) once leads to

$$(k + \frac{R_1}{k} - \frac{l\lambda^2}{k^2})u + \frac{(\lambda l + 2kl)}{2k} u^2 - \frac{l}{2} u^3 + k^2 l u'' = R_2. \tag{3.8}$$

where R_2 is an integration constant. Rewrite this second-order ordinary differential equation as follows

$$u''(\xi) + \frac{(k^3 + R_1 k - l\lambda^2)}{k^4 l} u(\xi) + \frac{(\lambda + 2k)}{2k^3} u^2(\xi) - \frac{1}{2k^2} u^3(\xi) - \frac{R_2}{k^2 l} = 0. \tag{3.9}$$

Using (5) and (6), we can get

$$\dot{X}(\xi) = Y(\xi), \tag{3.10}$$
$$\dot{Y}(\xi) = \frac{1}{2k^2} X^3(\xi) - \frac{(\lambda + 2k)}{2k^3} X^2(\xi) - \frac{(k^3 + R_1 k - l\lambda^2)}{k^4 l} X(\xi) + \frac{R_2}{k^2 l}. \tag{3.11}$$

According to the first integral method, we suppose the $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (16)-(17), and $Q(X, Y) = \sum_{i=0}^{m} a_i(X) Y^i = 0$ is an irreducible polynomial in the complex domain $C[X, Y]$ such that

$$Q(X(\xi), Y(\xi)) = \sum_{i=0}^{m} a_i(X(\xi)) Y^i(\xi) = 0, \tag{3.12}$$

where $a_i(X) (i = 0, 1, ..., m)$, are polynomials of X and $a_m(X) \neq 0$. Equation (18) is called the first integral to (16)-(17). Due to the Division Theorem, there exists a polynomial $g(X) + h(X)Y$, in the complex domain $C[X, Y]$ such that

$$\frac{dQ}{d\xi} = \frac{dQ}{dX}\frac{dX}{d\xi} + \frac{dQ}{dY}\frac{dY}{d\xi} = (g(X) + h(X)Y) \sum_{i=0}^{m} a_i(X) Y^i. \tag{3.13}$$

In this example, we take two different cases, assuming that $m = 1$ and $m = 2$ in (18).
Case A:
Suppose that $m = 1$, by comparing with the coefficients of $Y^i (i = 2, 1, 0)$ on both sides of (19), we have

$$\dot{a_1}(X) = h(X)a_1(X), \tag{3.14}$$
$$\dot{a_0}(X) = g(X)a_1(X) + h(X)a_0(X), \tag{3.15}$$

$$a_1(X)[\frac{1}{2k^2}X^3 - \frac{(\lambda+2k)}{2k^3}X^2 - \frac{(k^3+R_1k-l\lambda^2)}{k^4l}X + \frac{R_2}{k^2l}] = g(X)a_0(X). \tag{3.16}$$

Since $a_i(X)$ $(i = 0, 1)$ are polynomials, then from (20) we deduce that $a_1(X)$ is constant and $h(X) = 0$. For simplicity, take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that deg$(g(X)) = 1$ only. Suppose that $g(X) = A_1X + A_0$, then we find $a_0(X)$,

$$a_0(X) = c_1 + A_0X + \frac{1}{2}A_1X^2, \tag{3.17}$$

where c_1 is arbitrary integration constant.
Substituting $a_0(X)$ and $g(X)$ into (22) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$A_0 = -\frac{\lambda+2k}{3k^2},\ \ A_1 = \frac{1}{k},\ \ R_1 = -\frac{9c_1k^3l - 8l\lambda^2 + 4lk\lambda + 4k^2l + 9k^3}{9k},\ \ R_2 = -\frac{lc_1(\lambda+2k)}{3}, \tag{3.18}$$

$$A_0 = \frac{\lambda+2k}{3k^2},\ \ A_1 = -\frac{1}{k},\ \ R_1 = \frac{9c_1k^3l + 8l\lambda^2 - 4lk\lambda - 4k^2l - 9k^3}{9k},\ \ R_2 = \frac{lc_1(\lambda+2k)}{3}, \tag{3.19}$$

where k, l, λ and c_1 are arbitrary constants.
Using the conditions (24) in (18), we obtain

$$Y_1(\xi) = -c_1 + \frac{(\lambda+2k)}{3k^2}X(\xi) - \frac{1}{2k}X^2(\xi). \tag{3.20}$$

Combining (26) with (16), we obtain the exact solution to equations (16)-(17):

$$f_1(\xi) = \frac{(\lambda+2k)}{3k} - \frac{\sqrt{18c_1k^3 - \lambda^2 - 4k\lambda - 4k^2}}{3k}tan(\frac{\sqrt{18c_1k^3 - \lambda^2 - 4k\lambda - 4k^2}}{6k^2}(\xi+\xi_0)) \tag{3.21}$$

and the exact solution to $(2+1)-$dimensional dispersive long wave equation can be written as:

$$u_1(x,y,t) = \frac{(\lambda+2k)}{3k} - \frac{\sqrt{18c_1k^3 - \lambda^2 - 4k\lambda - 4k^2}}{3k} \times tan(\frac{\sqrt{18c_1k^3 - \lambda^2 - 4k\lambda - 4k^2}}{6k^2}(kx + ly - \lambda t + \xi_0)), \tag{3.22}$$

$$v_1(x,y,t) = -\frac{(9c_1k^3l - 8l\lambda^2 + 4l\lambda k + 4k^2l + 9k^3)}{9k^3} + \frac{\lambda l(\lambda+2k)}{3k^3} - \frac{l(\lambda+2k)^2}{18k^3}$$

$$
\begin{aligned}
&+\ \frac{2l(k-\lambda)\sqrt{18c_1k^3-\lambda^2-4k\lambda-4k^2}}{9k^3}\\
&\times\ tan(\frac{\sqrt{18c_1k^3-\lambda^2-4k\lambda-4k^2}}{6k^2}(kx+ly-\lambda t+\xi_0))\\
&-\ \frac{l(18c_1k^3-\lambda^2-4k\lambda-4k^2)}{18k^3}\\
&\times\ tan^2(\frac{\sqrt{18c_1k^3-\lambda^2-4k\lambda-4k^2}}{6k^2}(kx+ly-\lambda t+\xi_0)),
\end{aligned}
$$

where ξ_0 is an arbitrary constant.
Similarly, in the case of (25), from (18), we obtain

$$
Y_2(\xi)=-c_1-\frac{(\lambda+2k)}{3k^2}X(\xi)+\frac{1}{2k}X^2(\xi). \tag{3.23}
$$

and then the exact solution of $(2+1)-$dimensional dispersive long wave equation can be written as:

$$
\begin{aligned}
u_2(x,y,t)\ =&\ \frac{(\lambda+2k)}{3k}-\frac{\sqrt{18c_1k^3+\lambda^2+4k\lambda+4k^2}}{3k}\\
&\times\ tanh(\frac{\sqrt{18c_1k^3+\lambda^2+4k\lambda+4k^2}}{6k^2}(kx+ly-\lambda t+\xi_0)),
\end{aligned} \tag{3.24}
$$

$$
\begin{aligned}
v_2(x,y,t)\ =&\ \frac{(9c_1k^3l+8l\lambda^2-4l\lambda k-4k^2l-9k^3)}{9k^3}+\frac{\lambda l(\lambda+2k)}{3k^3}-\frac{l(\lambda+2k)^2}{18k^3}\\
&+\ \frac{2l(k-\lambda)\sqrt{18c_1k^3+\lambda^2+4k\lambda+4k^2}}{9k^3}\\
&\times\ tanh(\frac{\sqrt{18c_1k^3+\lambda^2+4k\lambda+4k^2}}{6k^2}(kx+ly-\lambda t+\xi_0))\\
&-\ \frac{l(18c_1k^3+\lambda^2+4k\lambda+4k^2)}{18k^3}\\
&\times\ tanh^2(\frac{\sqrt{18c_1k^3+\lambda^2+4k\lambda+4k^2}}{6k^2}(kx+ly-\lambda t+\xi_0)),
\end{aligned}
$$

where ξ_0 is an arbitrary constant.
Case B:
Suppose that $m=2$, by equating the coefficients of $Y^i(i=3,2,1,0)$ on both sides of (19), we have

$$
\dot{a_2}(X)\ =\ h(X)a_2(X), \tag{3.25}
$$

$$
\dot{a_1}(X)\ =\ g(X)a_2(X)+h(X)a_1(X), \tag{3.26}
$$

$$
\begin{aligned}
\dot{a_0}(X)\ =&\ -2a_2(X)[\frac{1}{2k^2}X^3-\frac{(\lambda+2k)}{2k^3}X^2-\frac{(k^3+R_1k-l\lambda^2)}{k^4l}X+\frac{R_2}{k^2l}]\\
&+\ g(X)a_1(X)+h(X)a_0(X),
\end{aligned} \tag{3.27}
$$

$$
a_1(X)[\frac{1}{2k^2}X^3-\frac{(\lambda+2k)}{2k^3}X^2-\frac{(k^3+R_1k-l\lambda^2)}{k^4l}X+\frac{R_2}{k^2l}]=g(X)a_0(X). \tag{3.28}
$$

Since $a_i(X)$ $(i=0,1,2)$ are polynomials, then from (31) we deduce that $a_2(X)$ is constant and $h(X)=0$. For simplicity, take $a_2(X)=1$. Balancing the degrees of $g(X),\ a_1(X)$ and $a_2(X)$, we conclude that deg$(g(X))=1$ only. Suppose that $g(X)=A_1X+B_0$, then we find $a_1(X)$ and $a_0(X)$ as follows

$$
a_1(X)=A_0+B_0X+\frac{1}{2}A_1X^2, \tag{3.29}
$$

$$a_0(X) = d + \frac{1}{4}(-\frac{1}{k^2} + \frac{A_1^2}{2})X^4 + \frac{1}{3}(\frac{3}{2}B_0A_1 + \frac{(\lambda+2k)}{k^3})X^3 + \frac{1}{2}(B_0^2 + A_0A_1 + \frac{2(k^3 + R_1k - l\lambda^2)}{k^4l})X^2 + (B_0A_0 - \frac{2R_2}{k^2l})X \quad (3.30)$$

Substituting $a_0(X), a_1(X)$ and $g(X)$ in the last equation in (34) and setting all the coefficients of powers X to be zero, then we obtain a system of nonlinear algebraic equations and by solving it with aid Maple, we obtain

$$d = \frac{A_0^2}{4},\ k = \frac{2}{A_1},\ \lambda = -\frac{2(3B_0 + 2A_1)}{A_1^2},\ R_2 = \frac{B_0lA_0}{A_1^2}, \quad (3.31)$$
$$R_1 = -\frac{2(-8B_0^2l - 12B_0lA_1 - 4lA_1^2 + A_0A_1l + 2A_1)}{A_1^3},$$

$$d = \frac{A_0^2}{4},\ k = -\frac{2}{A_1},\ \lambda = \frac{2(3B_0 + 2A_1)}{A_1^2},\ R_2 = \frac{B_0lA_0}{A_1^2}, \quad (3.32)$$
$$R_1 = \frac{2(-8B_0^2l - 12B_0lA_1 - 4lA_1^2 + A_0A_1l - 2A_1)}{A_1^3},$$

with B_0, A_0, A_1 and l are arbitrary constants.
Using the conditions (37) into (18), we get

$$Y_3(\xi) = -\frac{A_0}{2} - \frac{B_0}{2}X(\xi) - \frac{A_1}{4}X^2(\xi). \quad (3.33)$$

Combining (39) with (16), we obtain the exact solution to equations $(16) - (17)$ and the exact solution to $(2+1)-$dimensional dispersive long wave equation can be written as:

$$u_3(x,y,t) = -\frac{B_0}{A_1} - \frac{\sqrt{2A_0A_1 - B_0^2}}{A_1}tan(\frac{\sqrt{2A_0A_1 - B_0^2}}{4}(\xi + \xi_0)), \quad (3.34)$$

$$v_3(x,y,t) = -\frac{-8lB_0^2 - 12lB_0A_1 - 4lA_1^2 + lA_0A_1 + 2A_1}{A_1} + \frac{lB_0(3B_0 + 2A_1)}{2A_1} - \frac{lB_0^2}{4A_1} + \frac{l(B_0 + A_1)\sqrt{2A_0A_1 - B_0^2}}{A_1}tan(\frac{\sqrt{2A_0A_1 - B_0^2}}{4}(\xi + \xi_0)) - \frac{l(2A_0A_1 - B_0^2)}{4A_1}tan^2(\frac{\sqrt{2A_0A_1 - B_0^2}}{4}(\xi + \xi_0)),$$

where $\xi = \frac{2}{A_1}x + ly + \frac{2(3B_0+2A_1)}{A_1^2}t$ and ξ_0 is an arbitrary constant.
Similarly, in the case of (38), from (18), we obtain

$$Y_3(\xi) = -\frac{A_0}{2} - \frac{B_0}{2}X(\xi) - \frac{A_1}{4}X^2(\xi). \quad (3.35)$$

and then the exact solution to $(2+1)-$dimensional dispersive long wave equation can be written as:

$$u_4(x,y,t) = -\frac{B_0}{A_1} - \frac{\sqrt{2A_0A_1 - B_0^2}}{A_1}tan(\frac{\sqrt{2A_0A_1 - B_0^2}}{4}(\xi + \xi_0)), \quad (3.36)$$

$$v_4(x,y,t) = \frac{-8lB_0^2 - 12lB_0A_1 - 4lA_1^2 + lA_0A_1 - 2A_1}{A_1} + \frac{lB_0(3B_0 + 2A_1)}{2A_1} - \frac{lB_0^2}{4A_1}$$

$$- \frac{l(B_0 + A_1)\sqrt{2A_0A_1 - B_0^2}}{A_1} tan(\frac{\sqrt{2A_0A_1 - B_0^2}}{4}(\xi + \xi_0))$$
$$+ \frac{l(2A_0A_1 - B_0^2)}{4A_1} tan^2(\frac{\sqrt{2A_0A_1 - B_0^2}}{4}(\xi + \xi_0)),$$

where $\xi = -\frac{2}{A_1}x + ly - \frac{2(3B_0+2A_1)}{A_1^2}t$ and ξ_0 is an arbitrary constant.

4. Conclusion

In this work the first integral method was applied successfully for solving the $(2 + 1)-$dimensional dispersive long wave equation, Thus, we conclude that the proposed method can be extended to solve the nonlinear problems which arise in the theory of solitons and other areas.

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