International Journal of Applied Mathematics and Computation
Vol. 5 No. 1 (June, 2019)

KALAHARI JOURNALS

Received: 23rd July 2018 Revised: 14th December 2018 Accepted: 15th January 2019

Services and nature conservation in protected areas: An optimal control approach

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ABSTRACT

Protected Areas have to conserve the landscape and wildlife, allow people to visit and enjoy the area and help support local people. These different aims can sometimes conflict with each other, and tourism is one of the biggest challenges in Protected Areas, as tourists have both positive and negative impacts on the landscape and local communities. The administration of Protected Areas works with local communities and other organizations to try and make tourism more sustainable. We are interested in protected areas where there is no expected entrance fee, and thus the management of these areas is not self-funded. The paper is an attempt to optimize the allocation of government funding among the services requested by the visitor and the conservation of environmental resources. To achieve this goal, we model an optimal control problem, where we maximize the profit function, defined as the difference between the revenue from visitors and total investment in recreation, service and defensive expenditure for ensuring the preservation of the natural resource.

Keywords: Bioeconomic model; tourism; optimal dynamic control model; optimal policy mix.

1. Introduction

Protected areas (PAs), especially in the developing world, are critically under-funded (see [15] and [14]). For most parks, the dream of financial self-sufficiency remains elusive, and revenues from fees, ecological services and tourism are unlikely to catch ever-rising costs. To make things worse, the direct costs of operating protected areas are often dwarfed by the real or perceived indirect costs to surrounding communities; costs which have to be compensated in some way if the park is to succeed in the long term (see [8] and [7]). Funding from conventional sources such as governments, royalties, donors, etc. to conventional agents such as NGOs, park authorities, and agencies is perceived as unreliable, rarely sufficient, and under subject to strong competition. In times of reduced government spending or economic hardship when parks draw fewer paying visitors, philanthropic donations often make up an increasing portion of the area' budgets, but support from the government makes up the largest portion of funding for PA.[1] We consider a natural resource such as a protected area which has the potential to generate revenue through activities such as tourism, hunting, hiking and fishing. It is assumed that these activities cause some degree of environmental damage in the PA (see [1], [2] and [11]). Protected Area Management (PAM) is delegated to a manager who can allocate effort (time and resources) to a number of activities, including (i) promoting tourist activities and providing services to visitors; (ii) activities which protect

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[1]Unites States' national park government funding runs into billions of US dollars annually. Money from the government is usually broken down into two types of use: discretionary spending and mandatory spending. Discretionary spending covers normal park operations and special events. Mandatory spending goes towards programmess created and mandated by specific legislation.

and regenerate the resource (such as the control of weeds, feral animals, poaching, logging, etc.). We develop a optimal control model, where the focus is on optimality conditions for services and nature protection provided by the PAM. We manage to solve the model analytically for the long run, i.e. when the planning horizon is infinite. The paper is organized as follows. In Section 2 the model is developed, motivated by the perspective on the interactive development of infrastructure and resource protect. In Section 3 the optimal control policy is presented, with the analysis of dynamic behaviour, steady state and therefore a comparative static analysis is illustrated.

2. Structure of the model

2.1 *The park management*

We begin by considering the case where management of the PA is delegated to a public sector employee, who invests effort and available resources in two areas.[2] First, the PAM may invest in visitor services $(I_k(t))$. Activities in this category might include effort and resources spent on providing, administering and monitoring the quantity and quality of tourist services provided in the PA. Second, effort may be invested in activities that protect and regenerate the biological resource $(I_d(t))$. These might include tasks such as the control of invasive species (e.g., pests and weeds), poaching and illegal logging. Assuming a balanced budget, we have:

$$I = I_d(t) + I_k(t) \tag{2.1}$$

At each instant only one out of two policy instruments $(I_d(t), I_k(t))$ can be set independently. Similarly, we can write (2.1), as:

$$I_d(t) = \varphi(t)I(t) \tag{2.2}$$
$$I_k(t) = (1 - \varphi(t))I(t) \tag{2.3}$$

where $0 \leq \varphi(t) < 1$ is the fraction of public revenue used to finance cleanup and $0 < 1 - \varphi(t) \leq 1$ is the fraction of the investment in infrastructure. Thus, at each instant, policy can be summarized by $\varphi(t)$. Let $C(I_d(t), I_k(t))$ be the cost function increasing and convex in $I_d(t)$ and $I_k(t)$. The PAM benefit is given by

$$\Pi(V(t), I_d(t), I_k(t)) = pV(t) - C(I_d(t), I_k(t)) \tag{2.4}$$

where $V(t)$ is the number of visitors. The problem faced by the PAM then is to find dynamic interventions $I_d(t)$, $I_k(t)$ that maximize the present value of a continuous time-stream of revenues and is given by :

$$J = \max_{I_d(t), I_k(t)} \int_0^{\infty} (pV(t) - C(I_d(t), I_k(t)))e^{-\rho t} dt \tag{2.5}$$

where ρ is the time preference, which is given exogenously and is constant over time.

2.2 *The dynamic of natural resource $E(t)$*

We interpret environmental quality as a renewable resource. The quality of the environment accumulates owing to the regenerative capacity of nature that depends on the level of environmental quality. We

[2] In [9] are considered three activities: *i*) effort to the provision of visitor services; *ii*) effort in activities that protect and regenerate the biological resource and *iii*) consume leisure.

consider that tourism activity has damaging effects on the environment. In [10] is given an account of the environmental impacts of tourism such as energy consumption, water consumption, wastes, impact on water and air quality, ecosystem alteration and fragmentation, impact on wildlife, and on the aesthetic and cultural environment. The intensity of those impacts are closely related to the number of visitors and the building of facilities for their lodging and recreational activities. We assume that environmental quality evolves over time according to the following

$$\dot{E}(t) = \epsilon(\overline{E} - E(t)) - \alpha V(t) - \beta K(t) + \gamma I_d(t) \tag{2.6}$$

For simplicity we have considered a linear regeneration function (see [6]). $\overline{E}$ is the maximum level of environmental quality, $\epsilon > 0$ is the rate of recovery of the environment as a result of natural regeneration, $\alpha > 0$ measures the environmental impact associated with a unit of visitors, β is a technology parameter that quantifies the detrimental effect of infrastructure on the natural resource, and $0 < \gamma < 1$ is the technology parameter that measures the effectiveness of protection in the natural resource policy. Given this specification, investment in capacity has a negative impact on the environment, but investment in quality (higher capital for a given capacity of accommodation) does not. We do not differentiate between the environmental impact of different types of tourism. For instance, the differences in habits and behaviour of tourists with different socio-economic characteristics may imply differences in their environmental impact. Therefore, a change from mass tourism to quality tourism would not only affect the environment through the amount of tourists (assumedly in a positive way), but also from a change in α. A constant α is therefore a simplification only justified by our lack of evidence about the magnitude and even the sign of change in α when the composition of visitors changes.

2.3 *The dynamic of capital stock $K(t)$*

The investment $I_k(t)$ is used to increase the capital stock

$$\dot{K}(t) = I_k(t) - \kappa K(t) \tag{2.7}$$

Capital stock is assumed to depreciate at the rate $\kappa > 0$

2.4 *The dynamic of visitors $V(t)$*

Visitors are attracted by the infrastructures and services included in the variable $K(t)$ and by the natural resource $E(t)$. [3] Both stocks are combined by means of an additive[4] function, which assumes a degree of substitution between the environmental resource and the capital, in the sense that destinations with low capital stock can receive the same number of visitors as those with better infrastructures if they have a large natural resource stock. Dynamics of the number of visitors is (see, [9])

$$\dot{V}(t) = \sigma_1 E(t) + \sigma_2 K(t) - \nu V(t) \tag{2.8}$$

where $\sigma_1 > 0$ and $\sigma_2 > 0$ represent the preferences of visitors and the parameter $\nu > 0$ represents the crowding effect. This means that the PA becomes less attractive when the number of tourists visiting the protected area increases, and this gives rise to a decrease in the number of visitors.

[3] This assumption captures in a simple way the notion that biological resources are the key attraction in the park, but that the willingness to pay for the experience is also likely to depend on the available facilities such as the quality of accommodation, service, tour guides, transport facilities, etc.

[4] The results do not change if we use a multiplicative function such as σEK

3. Solution of Optimal Control Problem

In this section, we approach the problem from an optimal control point-of-view. From Section 2.1 and considering the investment $I(t) = I$ as constant and exogenous given, we can define the quadratic cost function as (i.e. [5])

$$C(\varphi(t)) = \frac{1}{2}\left(q_1 I_k^2(t) + q_2 I_d^2(t)\right) = \frac{I^2}{2}\left(q_1\varphi(t)^2 + q_2(1-\varphi(t))^2\right) \tag{3.1}$$

Then the optimal control problem with infinite time horizon becomes

$$J = \max_{\varphi(t)} \int_0^\infty (pV(t) - \frac{I^2}{2}(q_1\varphi(t)^2 + q_2(1-\varphi(t))^2))e^{-\rho t}dt$$

subject to

$$\begin{aligned}
\dot{E}(t) &= \epsilon(\overline{E} - E(t)) - \alpha V(t) - \beta K(t) + \gamma\varphi(t)I \\
\dot{K}(t) &= (1-\varphi(t))I - \kappa K(t) \\
\dot{V}(t) &= \sigma_1 E(t) + \sigma_2 K(t) - \nu V(t) \\
\varphi(t) &\in [0,1] \\
E(0) &= E_0 \\
K(0) &= K_0 \\
V(0) &= V_0 \\
t &\in [0,\infty).
\end{aligned} \tag{3.2}$$

An approach to find a solution of the problem is based on the Pontryagin Maximum Principle for a case of infinite time horizon. More precisely, we use *Corollary 7* in [4]. First, let us check that the problem satisfied a number of assumptions in order to prove the applicability of the method for the problem. For each E, K and V, the function $\Pi(V(t), \varphi(t))$ is a concave function in $\varphi(t)$ That follows from convexity of the function $C(\varphi(t))$. There exist positive valued functions $\mu(t)$ and ω on $[0,\infty)$ such that $\mu(t) \to 0$ as $t \to \infty$ and for any admissible pair (V, φ),

$$e^{\rho t} \max_{\varphi(t)\in[]0,1]} |\Pi(V(t), \varphi(t)| \le \mu(t) \quad \text{for all } t > 0;$$

$$\int_0^\infty e^{-\rho t}|\Pi(V(t), \varphi(t))|dt \le \omega(T) \quad \text{for all } T > 0$$

This follows on the linearity of the function $\Pi(V, \varphi)$ in V, and the restrictions on the control function. There exists a $h \ge 0$ and a $r \ge 0$ such that

$$\frac{\partial \Pi(V,\varphi)}{\partial V} \le h(1+V)^r \quad \text{for all } V \text{ and for all } \varphi \in [0,1]\ .$$

Taking into account the linearity of the function $\Pi(V, \varphi)$ in V, we get $h = p$ and $r = 0$.

$$\rho > (r+1)\lambda,$$

where λ is the maximal of the real parts of the eigenvalues of the dynamic system. If $r = 0$ and all eigenvalues of the dynamic system are negative, it is sufficient that $\rho > 0$.

Now we can start solving the problem using the Maximum Principle. Let us compose the current Hamiltonian- Pontryagin function

$$\mathcal{H} = pV - \frac{I^2}{2}(q_1\varphi^2 + q_2(1-\varphi)^2)) + \lambda_E \dot{E} + \lambda_K \dot{K} + \lambda_V \dot{V} \tag{3.3}$$

where the terms λ_E, λ_K, λ_V are the co-state variables which can be interpreted as the shadow value, measured in utility terms, of natural resource E of capital stock K and of visitors V, respectively. We get first order conditions (FCO) for a maximum by taking the derivatives with respect to control φ and state (i.e. E, K, V) variables. Denoting a derivative with respect to one of the control or state variables by the corresponding subscript, we get the following equations:

$$\mathcal{H}_\varphi = 0 \Leftrightarrow \varphi = \frac{q_2 I + \gamma\lambda_E - \lambda_K}{(q_1+q_2)I} \tag{3.4}$$

$$\mathcal{H}_E = \rho\lambda_E - \dot{\lambda}_E \tag{3.5}$$

$$\mathcal{H}_K = \rho\lambda_K - \dot{\lambda}_K \tag{3.6}$$

$$\mathcal{H}_V = \rho\lambda_V - \dot{\lambda}_V \tag{3.7}$$

Substituting (3.4) in (3.2) and by straightforward calculation, together with differential equations (3.5)-(3.7), we obtain a set of six linear differential equations determining the linear dynamic system.[5]

$$\dot{E} = \epsilon(\overline{E} - E) - \alpha V - \beta K + \gamma\frac{q_2 I + \gamma\lambda_E - \lambda_K}{(q_1+q_2)} \tag{3.8}$$

$$\dot{K} = \frac{q_1 I - \gamma\lambda_E + \lambda_K}{(q_1+q_2)} - \kappa K \tag{3.9}$$

$$\dot{V} = \sigma_1 E + \sigma_2 K - \nu V \tag{3.10}$$

$$\dot{\lambda}_E = (\rho+\epsilon)\lambda_E - \sigma_1\lambda_V \tag{3.11}$$

$$\dot{\lambda}_K = (\rho+\kappa)\lambda_K + \beta\lambda_E - \sigma_2\lambda_V \tag{3.12}$$

$$\dot{\lambda}_V = (\rho+\nu)\lambda_V + \alpha\lambda_E - p \tag{3.13}$$

Investigating the signs (3.11), (3.12) and (3.13) for $\lambda_E > 0$, $\lambda_K > 0$ and $\lambda_V > 0$ shows that positive and negative factors emerge. Thus, the costate variables decrease or increase over time, depending on the relative forces of the weights of the rate of recovery of environmental, depreciation of capital, crowding effect and attraction of visitors.

The transversality condition

$$\lim_{t\to\infty} e^{-rt}\lambda_K(t)K(t) = 0 \tag{3.14}$$

$$\lim_{t\to\infty} e^{-rt}\lambda_V(t)V(t) = 0 \tag{3.15}$$

$$\lim_{t\to\infty} e^{-rt}\lambda_E(t)E(t) = 0 \tag{3.16}$$

allows us to derive values for the control variables for $t \to \infty$. The first-order conditions, together with the transversality condition are, in this model, sufficient for a maximum. In fact, the *maximized Hamiltonian* H^0 is concave (being sum of concave functions) in the state variables (see [3]).

[5] In general, linearized models are often used in macroeconomics as approximations (usually in the neighborhood of an equilibrium) to non-linear models (see [12] and [13]).

3.1 *Analysis of dynamic system*

As first step we analyse the steady state of the model and present some results of a comparative static analysis of the steady state with respect to important parameters of the dynamic system. In the following we investigate the behaviour of our model outside the steady state.
Define:
$\Phi := \dfrac{\gamma\lambda_E - \lambda_K}{q_1 + q_2}$
$q := \dfrac{q_2}{q_1 + q_2}$
then $\varphi = q + \dfrac{\Phi}{I}$ and from straightforward calculation, we can say that:

Proposition 3.1. *If*

a) $\Phi \geq 0 \Rightarrow \dfrac{\lambda_K}{\lambda_E} \leq \gamma$ *then* $\varphi > 0$ *is always held, while* $\varphi \leq 1$ *iff.* $I \geq \dfrac{\Phi}{1-q}$

b) $\Phi < 0 \Rightarrow \dfrac{\lambda_K}{\lambda_E} > \gamma$ *then* $\varphi < 1$ *is always hold, while* $\varphi \geq 0$ *iff.* $I > \dfrac{|\Phi|}{q}$

3.2 *Steady State*

We focus on the long-run. By setting $\dot{E} = \dot{K} = \dot{V} = \dot{\lambda_E} = \dot{\lambda_K} = \dot{\lambda_V} = 0$ in (3.8)-(3.13), one gets the unique solution $S = (E^*, K^*, V^*, \lambda_E^*, \lambda_K^*, \lambda_V^*)$.
We define:
$\varphi^* := q + \dfrac{\Phi^*}{I}$
$\Gamma := \dfrac{\frac{\gamma}{\alpha}q - (1-q)(\frac{\sigma_2}{\nu\kappa} + \frac{\beta}{\alpha\kappa})}{\frac{\sigma_1}{\nu} + \frac{\epsilon}{\alpha}}$
$\Lambda := \dfrac{\frac{\gamma}{\alpha} + \frac{\sigma_2}{\nu\kappa} + \frac{\beta}{\alpha\kappa}}{\frac{\sigma_1}{\nu} + \frac{\epsilon}{\alpha}}$
$\Omega := \dfrac{\frac{\epsilon}{\alpha}}{\frac{\sigma_1}{\nu} + \frac{\epsilon}{\alpha}}$
$\overline{\sigma_2} := \dfrac{\nu}{\alpha}[\dfrac{\gamma k q}{\alpha(1-q)} - \beta]$
where $\Phi^* = \dfrac{\gamma\lambda_E^* - \lambda_K^*}{q_1 + q_2}$ and λ_E^* e λ_K^* are obtained by solving the sub system $\dot{\lambda}_E = \dot{\lambda}_K = \dot{\lambda}_V = 0$. Therefore, we can write the following proposition:

Proposition 3.2. *The steady state* $S_\varphi = (E^*, K^*, V^*, \lambda_E^*, \lambda_K^*, \lambda_V^*, \varphi^*)$ *(with* $0 \leq \varphi^* \leq 1$*) exists iff. the following conditions are verified :*

$$\frac{\sigma_1}{\sigma_2} < \frac{\rho + \epsilon}{\beta}, \qquad \Gamma I + \Phi^* \Lambda + \Omega\overline{E} > 0 \tag{3.17}$$

in particular, if

a) $\Phi^* < 0 \Rightarrow \dfrac{\sigma_1}{\sigma_2} < \dfrac{\rho + \epsilon}{\beta + \gamma(\rho + \kappa)}$ *then*

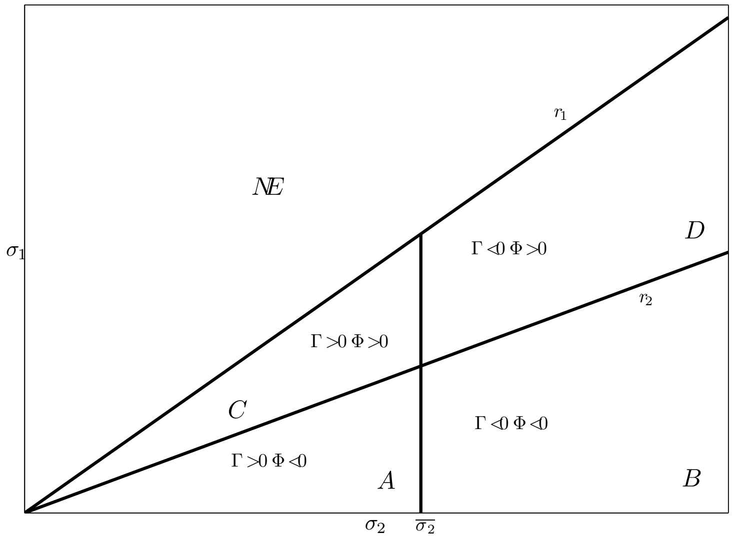


Figure 1: Diagram of existence of steady state in plane (σ_2, σ_1).

a.1) $\Gamma > 0$, *implies that steady state exists for* $I > max\{\frac{|\Phi^*|}{q}, \frac{|\Phi^*|\Lambda - \Omega\overline{E}}{\Gamma}\}$ *(Region A in Figure 1)*

a.2) $\Gamma < 0$, *implies that steady state exists for* $\frac{|\Phi^*|}{q} < I < \frac{-|\Phi^*|\Lambda + \Omega\overline{E}}{|\Gamma|}$ *(Region B in Figure 1)*

b) $\Phi^* > 0 \Rightarrow \frac{\sigma_1}{\sigma_2} > \frac{\rho + \epsilon}{\beta + \gamma(\rho + \kappa)}$ *then*

b.1) $\Gamma > 0$, *implies that steady state exists for* $I > \frac{\Phi^*}{1-q}$ *(Region C in Figure 1)*

b.2) $\Gamma < 0$, *implies that steady state exists for* $\frac{\Phi^*}{1-q} < I < \frac{\Phi^*\Lambda + \Omega\overline{E}}{|\Gamma|}$ *(Region D in Figure 1)*

Proof. From straightforward calculation the steady state $S = (E^*, K^*, V^*, \lambda_E^*, \lambda_K^*, \lambda_V^*)$ is

$$E^* = \Gamma I + \Phi^*\Lambda + \Omega\overline{E} \tag{3.18}$$

$$K^* = \frac{1-\varphi^*}{\kappa} I = (1-q)\frac{I}{\kappa} - \frac{\Phi^*}{\kappa} \tag{3.19}$$

$$V^* = (\frac{\sigma_1}{\nu}\Gamma + \frac{\sigma_2}{\nu\kappa}(1-q))I + \frac{\Phi^*}{\nu}(\sigma_1\Lambda - \frac{\sigma_2}{\kappa}) + \frac{\sigma_1}{\nu}\Omega\overline{E} \tag{3.20}$$

$$\lambda_E^* = \frac{p\sigma_1}{\alpha\sigma_1 + (\rho+\epsilon)(\rho+\nu)} \tag{3.21}$$

$$\lambda_K^* = \frac{p(\sigma_2(\rho+\epsilon) - \beta\sigma_1)}{(\rho+\kappa)(\alpha\sigma_1 + (\rho+\nu)(\rho+\epsilon))} \tag{3.22}$$

$$\lambda_V^* = \frac{p(\rho+\epsilon)}{\alpha\sigma_1 + (\rho+\nu)(\rho+\epsilon)} \tag{3.23}$$

From easy calculation, the sign of $\Phi^* = \dfrac{\gamma\lambda_E^* - \lambda_K^*}{q_1 + q_2}$ is given by sign of $\sigma 1(\gamma(\rho+\kappa)+\beta) - \sigma_2(\rho+\epsilon)$. Therefore the inequality $\lambda_K^* > 0$ and $E^* > 0$ given the conditions (3.17), while the others conditions are give by the conditions of proposition 3.1 and $E^* > 0$. □

Proposition 3.2 highlights some interesting things. If $\Gamma > 0$ the effort I is limited conditioner under **a.1**) and **b.1**), any value above a certain threshold leads to the admissibility of the steady state S_φ and then to an optimal solution of the problem. The case $\Gamma < 0$ implies that I is also top bound, then on top of a certain value in steady state S_φ is not permissible. So it is interesting to note that, for certain values of parameters, the PAM has no reason to require that values of I exceed the thresholds set out in the conditions **a.2**) and **b.2**). This is justified by the fact that Γ, as shown easily from a comparative static analysis on E^* in function of the investment I, indicates the direction and value variation of the stock of resource environmental balance, so if its sign is negative we will have a decrease of this with increasing stocks' investment, until it reaches full deterioration of the resource. It is also useful to represent the necessary condition $\dfrac{\sigma_1}{\sigma_2} < \dfrac{\rho+\epsilon}{\beta} = r_1$ and $\dfrac{\sigma_1}{\sigma_2} < \dfrac{\rho+\epsilon}{\beta+\gamma(\rho+\kappa)} = r_2$ in plane (σ_2, σ_1), (set the parameters ρ, β, ϵ, γ, κ) as in Figure 1, where are highlighted areas characterized by the signs of Γ and Φ and region plan (NE) in which the steady state does not exist. It can be easily observed that the more economic agents is patience (ρ), and the higher the rate of regeneration environmental resources ϵ (ceteris paribus), the wider are the permitted visitors preferences. Indeed, increasing the slope of r_1, decreases so the region NE.

3.3 *Dynamic behavior and stability of the steady state*

Our first step in analysing the dynamic behaviour of linear model. To simplify notation we define:

$$\mathbf{z} := (E - E^*, K - K^*, V - V^*, \lambda_E - \lambda_E^*, \lambda_K - \lambda_K^*, \lambda_V - \lambda_V^*)^T \tag{3.24}$$

The vectors z measures the distance of each endogenous variable from its steady state value. Taking into account that $\dot{\mathbf{z}} = (\dot{E}, \dot{K}, \dot{V}, \dot{\lambda}_E, \dot{\lambda}_K, \dot{\lambda}_V)$, the system (3.8)-(3.13) is given the following vector-equation:

$$\dot{\mathbf{z}} = J\mathbf{z}. \tag{3.25}$$

J is the Jacobian matrix of the system of equation ((3.8))-((3.13)). Thus the general solution of system is determined by:

$$\mathbf{z}(t) = \mathbf{z}(0)e^{Jt} \tag{3.26}$$

Denoting the eigenvalues of J with μ_i, $i = 1..6$ and the six corresponding eigenvectors with $\mathbf{v}_i$, $i = 1..6$, we may rewrite the general solution as follow:

$$\mathbf{z}(t) = \sum_{i=1}^{6} a_i e^{\mu_i t}\mathbf{v}_i \tag{3.27}$$

where the scalar a_i is determined by the initial conditions $\mathbf{z}(0) = \mathbf{z}_0 = \sum_{i=1}^{6} a_i\mathbf{v}_i$. The vector space, which contains the solution of (3.25), may be divided in two subspaces. One of them is spanned by eigenvectors $\mathbf{v}_i$, which correspond to the negative eigenvalues (respectively, the complex eigenvalues with a negative real part). This is the stable subspace, because solutions in this subspace run into the steady state in the course of time. The other one is the unstable subspace, spanned by eigenvectors, which correspond to the positive eigenvalues (respectively, the complex eigenvalues with a positive real part). Therefore, we restrict the following argumentation on the case of three negative eigenvalues, i.e. the case where stable manifold exists. Taking into account the trasversality condition (3.14)-(3.16), we obtain

$$\lambda_E(t) = \lambda_E^* > 0, \qquad \lambda_K(t) = \lambda_K^* > 0, \qquad \lambda_V(t) = \lambda_V^* > 0 \tag{3.28}$$

and

$$\varphi^*(t) = q + \frac{\gamma\lambda_E^* - \lambda_K^*}{q_1 + q_2} \qquad \in (0,1] \tag{3.29}$$

Therefore an important conclusion is that the optional control $\varphi^*(t)$ is a constant function over time $t \in [0, \infty)$.
Let us substitute the constant control φ^* into the equation describing the dynamical system. The equations take the form

$$\dot{E} = \epsilon(\overline{E} - E) - \alpha V - \beta K + \gamma\varphi^* I \tag{3.30}$$

$$\dot{K} = (1 - \varphi^*)I - \kappa K \tag{3.31}$$

$$\dot{V} = \sigma_1 E + \sigma_2 K - \nu V \tag{3.32}$$

This means that, assuming $\varphi(t) = \varphi^*$, the trajectory of the system can be computed as the solution of the affine system

$$\dot{x} = Jx + G$$

where

$$x = \begin{pmatrix} E \\ K \\ V \end{pmatrix}, \qquad J = \begin{pmatrix} -\epsilon & -\beta & -\alpha \\ 0 & -\kappa & 0 \\ \sigma_1 & \sigma_2 & -\nu \end{pmatrix}, \qquad G = \begin{pmatrix} -\gamma\varphi^* I \\ -(1-\varphi^*)I \\ 0 \end{pmatrix}$$

satisfying the initial condition

$$x(0) = \begin{pmatrix} E(0) \\ K(0) \\ V(0) \end{pmatrix}$$

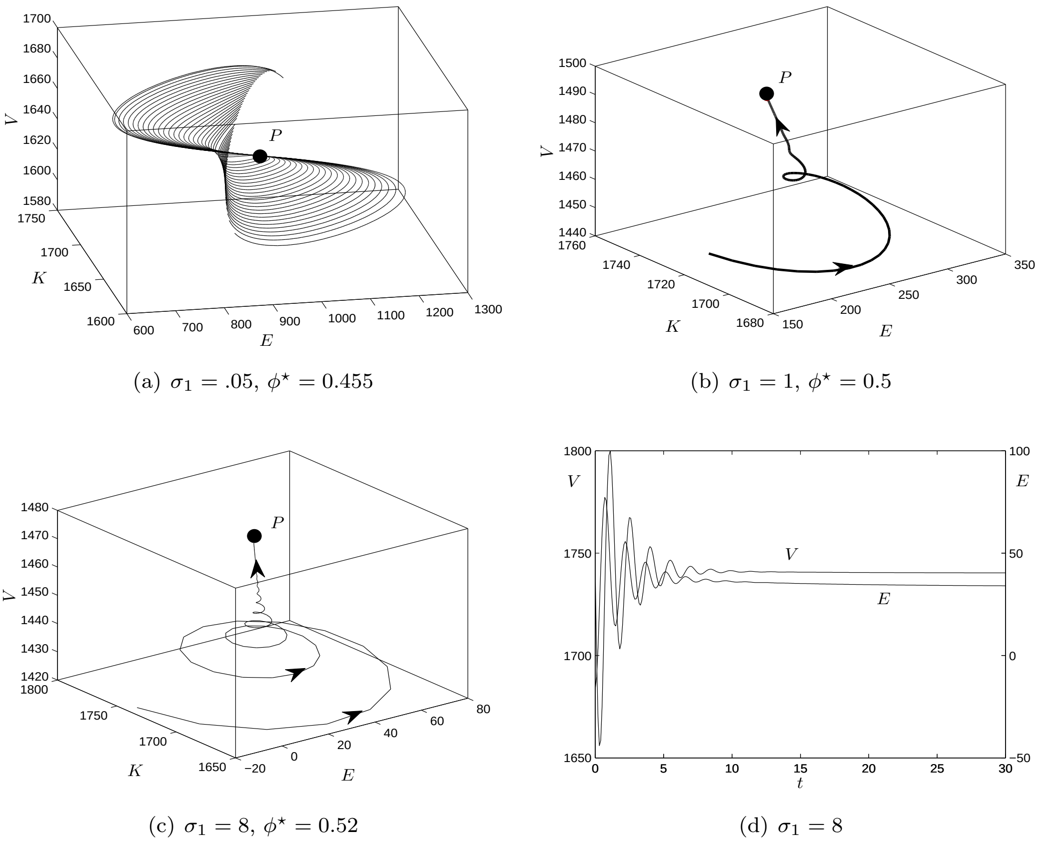
The solution to the system above can be calculated analytically as

$$\begin{pmatrix} E(t) \\ K(t) \\ V(t) \end{pmatrix} = c_1 \begin{pmatrix} \frac{-\mu_1+\nu}{\sigma_1} \\ 0 \\ 1 \end{pmatrix} e^{\mu_1 t} + c_2 \begin{pmatrix} \frac{-\mu_2+\nu}{\sigma_1} \\ 0 \\ 1 \end{pmatrix} e^{\mu_2 t} + c_3 \begin{pmatrix} a \\ b \\ 1 \end{pmatrix} e^{-kt} + \begin{pmatrix} E^* \\ K^* \\ V^* \end{pmatrix}$$

where c_i, a, b and the eigenvalues μ_i are defined in the Appendix. In our system we can say that:

Proposition 3.3. *The steady state S is always a saddle point, with three negative(possibly real part) and three positive (possibly real part) eigenvalues. Moreover if $(\nu - \epsilon)^2 - 4\alpha\sigma_1 > 0$ then the system has real eigenvalues otherwise it has complex eigenvalues.*

Proof. See Appendix



(a) $\sigma_1 = .05$, $\phi^\star = 0.455$

(b) $\sigma_1 = 1$, $\phi^\star = 0.5$

(c) $\sigma_1 = 8$, $\phi^\star = 0.52$

(d) $\sigma_1 = 8$

Figure 2: The parameter values are: $\alpha = 2.3$, $\beta = 0.02$, $\epsilon = 0.1$, $\gamma = 1$, $\rho = 0.1$, $\kappa = 0.1$, $\nu = 1$, $\sigma_2 = 1$, $q_1 = 1$, $q_2 = 1$, $\overline{E} = 40000$, $p = 10$, $I = 300$.

4. Numerical Simulations

In this section we present computer simulation of some solutions of the system (2). From practical point of view numerical solutions are very important beside analytical study. We take the parameters of the system as $\alpha = 2.3$,$\beta = 0.02$, $\epsilon = 0.1$, $\gamma = 1$, $\rho = 0.1$, $\kappa = 0.1$, $\nu = 1$,$\sigma_2 = 1$ $q_1 = 1$, $q_2 = 1$, $\overline{E} = 40000$, $p = 10$, $I = 300$. By this values of the parameter, for $\sigma_1 =< 0.1$, the eigenvalues are real, whereas they are complex. Figure 2 shows the phase diagrams of the system (3.2) to varying the parameter σ_1. Figure 2(a) illustrates, for $\sigma_1 = 0.05 < 0.1$, different paths that converge to steady state of coordinates $P = (E^\star = 952, K^\star = 1638, K^\star = 1685)$, whit $\phi^\star = 0.455$. Figures 2 (b)-(c) illustrate, increasing σ_1, the oscillatory behaviour of the variables $K(t)$, $E(t)$ and $V(t)$, which approach to steady state. In particular, Figure 2(d), shows how this oscillatory trajectory, can be very 'dangerous', leading the stock of natural resource to zero. As we observe the values of $\phi^\star$, we see that it increases with increasing σ_1. This can be explained, by recalling that the parameter σ_1 represents the preference of visitors towards the natural resource, as σ_1 increases the more it is necessary to defend the natural resource.

5. Conclusion

This article has studied the link between environmental resources and visitors in the Protected Areas. By emphasizing the optimal use of government funding required by the management of protected areas, the proposed model has shown that within an appropriate choice of parameters it is not convenient to have more funding (see, conditions a.2) and b.2 in Proposition 1). In addition, work has shown that for high values of the parameter sigma, you can have fluctuations of state variables, causing permanent damage to the natural resource and thus drastically reducing the number of visitors (see, Proposition 2 and Figure 2(d)).

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Appendix

$$\overline{c} = \sigma_1 \frac{E(0) - E^* - \dfrac{a}{b}(K(0) - K^*) - (V(0) - K(0) + K^* - V^*)}{-(\epsilon + \nu)}$$

$$c_1 = \overline{c} + \frac{\mu_2}{\epsilon + \mu}(V(0) - K(0) + K^* - V^*)$$

$$c_2 = -\bar{c} + \frac{\mu_1}{\epsilon + \mu}(V(0) - K(0) + K^* - V^*)$$
$$c_3 = \frac{K(0) - K^*}{b}$$
$$a = \frac{-\sigma 2\alpha + \kappa\beta - \nu\beta}{-\sigma_1\beta - \kappa\sigma_2 + \epsilon\sigma 2}$$
$$b = -\frac{\nu\kappa - \sigma_1\alpha + \kappa\epsilon - \kappa^2 - \nu\epsilon}{-\sigma_1\beta - \kappa\sigma_2 + \epsilon\sigma 2}$$

From the Jacobian J the eigenvalues are

$$\mu_{1,2} = -\frac{\epsilon}{2} - \frac{\nu}{2} \pm \frac{1}{2}\sqrt{(\epsilon - \nu)^2 - 4\alpha\sigma_1} \; \mu_3 = -\kappa$$