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Some results for the weighted Drazin inverse of a modified matrix*

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ABSTRACT

In this paper, we give some results for the W-weighted Drazin inverse of a modified matrix $M = A - CWD_{d,w}WB$ in terms of the W-weighted Drazin inverse of the matrix A and the generalized Schur complement $Z = D - BWA_{d,w}WC$, generalizing some recent results in the literature.

Keywords: Drazin inverse; Weighted Drazin inverse; Modified matrix; Schur complement

1. Introduction

Let $\mathbb{C}^{m\times n}$ denote the set of $m \times n$ complex matrices. The Drazin inverse of $A \in \mathbb{C}^{n\times n}$ is the unique matrix X, denoted by A_d, satisfying the following equations

$$A^{k+1}X = A^k,\ XAX = X,\ AX = XA, \tag{1.1}$$

where $k = ind(A)$ is the index of A, the smallest nonnegative integer for which $rank(A^{k+1}) = rank(A^k)$ (see[1-3]). In particular, when $ind(A) = 1$, the Drazin inverse of A is called the group inverse of A and is denoted by A_g. If A is nonsingular, it is clearly $ind(A) = 0$ and $A^D = A^{-1}$. Throughout this paper, we denote by $A^\pi = I - AA_d$ and define $A^0 = I$, where I is the identity matrix with proper sizes. In addition, the symbols $r(A)$ and $\|A\|$ will stand for rank and spectral norm of $A \in \mathbb{C}^{m\times n}$.

Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$ with $ind(AW) = k$ and $X \in \mathbb{C}^{m\times n}$ be a matrix such that

$$(AW)^{k+1}XW = (AW)^k,\ XWAWX = X,\ AWX = XWA, \tag{1.2}$$

then X is called W-weighed Drazin inverse of A and denoted by $X = A_{d,w}$ [4]. In particular, when A is square matrix and $W = I$ then $A_{d,w} = A_d$.

The importance of the Drazin inverse (W-weighted Drazin inverse) and its applications are very useful which can be found in [1-13]. In 2006, Hartwig et al. [5] gave some expressions for the Drazin inverse and the W-weighted Drazin inverse in order to find the solution of a second-order differential equation

$$Ex''(t) + Fx'(t) + Gx(t) = 0.$$

In 1975, Shoaf [6] found the result of the Drazin inverse of modified square matrix, in 1994, Radoslaw et al. [14] presented an explicit representation for the generalized inverse of a modified matrix, and in 2002, Wei [11]

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have discussed the expression of the Drazin inverse of a modified square matrix $A - CB$. Recently, in 2008, Xu et al. [15,16] gave some explicit expressions for the weighted Drazin inverse of a rectangular matrix $A - CB$ and $A - CWB$.

This paper is organized as follows. In section 2, we give some results for the W-weighted Drazin inverse of the modified matrix $M = A - CWD_{d,w}WB$ in terms of the W-weighted drazin inverse of the matrix A and the generalized Schur complement $Z = D - BWA_{d,w}WC$. Some relative results in [10,11,17] are the corollaries of our paper.

2. The W-weighted Drazin inverse of a modified matrix

In this section, we present some results for the W-weighted Drazin inverse of the modified matrix $M = A - CWD_{d,w}WB$ in terms of the W-weighted drazin inverse of the matrix A and the generalized Schur complement $Z = D - BWA_{d,w}WC$. As a result, some conclusions in [10,11,17] are obtained directly from our results.

Let $A,\ B,\ C,\ D \in \mathbb{C}^{m\times n},\ W \in \mathbb{C}^{n\times m}$. Throughout this paper, we adopt the following notations:

$$K = A_{d,w}WC,\ H = BWA_{d,w},\ \Gamma = HWK, \tag{2.1}$$

$$P = (I - AWA_{d,w}W)C,\ Q = B(I - WA_{d,w}WA). \tag{2.2}$$

Theorem 2.1. Suppose $P = 0,\ Q = 0,\ C(I - WD_{d,w}WD)WZ_{d,w}WB = 0,\ CWD_{d,w}W(I - ZWZ_{d,w}W)B = 0,\ C(I - WZ_{d,w}WZ)WD_{d,w}WB = 0$ and $CWZ_{d,w}W(I - DWD_{d,w}W)B = 0$, then

$$M_{d,w} = A_{d,w} + KWZ_{d,w}WH. \tag{2.3}$$

Proof. Let the right hand side of (2.3) be X. Since

$$\begin{aligned}
& MWX \\
=\ & (AW - CWD_{d,w}WBW)(A_{d,w} + KWZ_{d,w}WH) \\
=\ & AWA_{d,w} + AWKWZ_{d,w}WH - CWD_{d,w}WBWA_{d,w} \\
& -CWD_{d,w}WBWKWZ_{d,w}WH \\
=\ & AWA_{d,w} + CWZ_{d,w}WH - CWD_{d,w}WH \\
& -CWD_{d,w}W(D - Z)WZ_{d,w}WH \\
=\ & AWA_{d,w} + C(I - WD_{d,w}WD)WZ_{d,w}WBWA_{d,w} \\
& -CWD_{d,w}W(I - ZWZ_{d,w}W)BWA_{d,w} \\
=\ & AWA_{d,w}
\end{aligned}$$

and

$$\begin{aligned}
& XWM \\
=\ & (A_{d,w} + KWZ_{d,w}WH)(WA - WCWD_{d,w}WB) \\
=\ & A_{d,w}WA - A_{d,w}WCWD_{d,w}WB + KWZ_{d,w}WHWA \\
& -KWZ_{d,w}WHWCWD_{d,w}WB \\
=\ & A_{d,w}WA - KWD_{d,w}WB + KWZ_{d,w}WB \\
& -KWZ_{d,w}W(D - Z)WD_{d,w}WB \\
=\ & A_{d,w}WA - A_{d,w}WC(I - WZ_{d,w}WZ)WD_{d,w}WB \\
& +A_{d,w}WCWZ_{d,w}W(I - DWD_{d,w}W)B \\
=\ & A_{d,w}WA.
\end{aligned}$$

Thus

$$MWX = XWM. \tag{2.4}$$

While

$$\begin{aligned} XWMWX &= A_{d,w}WAW(A_{d,w} + KWZ_{d,w}WH) \\ &= A_{d,w}WAWA_{d,w} + A_{d,w}WAWKWZ_{d,w}WH \\ &= A_{d,w} + KWZ_{d,w}WH \\ &= X. \end{aligned}$$

Finally, by induction we will prove that

$$(MW)^{k+1}XW = (MW)^k, \tag{2.5}$$

where $k \geq l = Ind(AW)$. For the case $l = Ind(AW) = 1$, it is easy to see from $(AW)^2 A_{d,w}W = AW$ that

$$\begin{aligned} (MW)^2XW &= MWMWXW \\ &= (A - CWD_{d,w}WB)WAWA_{d,w}W \\ &= AW - CWD_{d,w}WBW \\ &= MW. \end{aligned}$$

Generally, for $l = Ind(AW) > 1$, note the fact $(AW)^{l+1}A_{d,w}W = (AW)^l$ that

$$\begin{aligned} &(MW)^{l+1}XW \\ =\ &(MW)^l MWXW \\ =\ &[(A - CWD_{d,w}WB)W]^l AWA_{d,w}W \\ =\ &(A - CWD_{d,w}WB)W(A - CWD_{d,w}WB)W \cdots \\ &\times(A - CWD_{d,w}WB)WAWA_{d,w}W \\ =\ &(I - CWD_{d,w}WBWA_{d,w}W)AW(I - CWD_{d,w}WBWA_{d,w}W)AW \cdots \\ &\times(I - CWD_{d,w}WBWA_{d,w}W)AWAWA_{d,w}W \\ =\ &(I - CWD_{d,w}WBWA_{d,w}W)[I - AWCWD_{d,w}WBW(A_{d,w}W)^2](AW)^2 \cdots \\ &\times(I - CWD_{d,w}WBWA_{d,w}W)AWAWA_{d,w}W \\ =\ &\cdots \\ =\ &(I - CWD_{d,w}WBWA_{d,w}W)[I - AWCWD_{d,w}WBW(A_{d,w}W)^2] \\ &\times[I - (AW)^2CWD_{d,w}WBW(A_{d,w}W)^3] \cdots \\ &\times[I - (AW)^{l-1}CWD_{d,w}WBW(A_{d,w}W)^l](AW)^{l+1}A_{d,w}W \\ =\ &(I - CWD_{d,w}WBWA_{d,w}W)[I - AWCWD_{d,w}WBW(A_{d,w}W)^2] \cdots \\ &\times[I - (AW)^{l-2}CWD_{d,w}WBW(A_{d,w}W)^{l-1}](AW)^{l-1} \\ &\times(AW - CWD_{d,w}WBWA_{d,w}WAW) \\ =\ &\cdots \\ =\ &(AW - CWD_{d,w}WBWA_{d,w}WAW)(AW - CWD_{d,w}WBWA_{d,w}WAW) \cdots \\ &\times(AW - CWD_{d,w}WBWA_{d,w}WAW) \\ =\ &[(A - CWD_{d,w}WB)W]^l \\ =\ &(MW)^l. \end{aligned}$$

For $k \geq l = Ind(AW)$, now we obtain that

$$(MW)^{k+1}XW = (MW)^k.$$

Therefore, (2.5) holds, which completes the proof. □

When $A,\ B,\ C,\ D$ are square and $W = I$ in our Theorem 2.1, we obtain Theorem 2.1 in [17] as a corollary of our Theorem 2.1.

Corollary 2.1 ([17]). Let $A,\ B,\ C,\ D \in \mathbb{C}^{m\times m}$, and $W = I$ in (2.1) and (2.2). Suppose $P = 0,\ \ Q = 0$, $C(I - DD_d)Z_dB = 0, CD_d(I - ZZ_d)B = 0,\ C(I - ZZ_d)D_dB = 0$ and $CZ_d(I - DD_d)B = 0$, then

$$M_d = A_d + KZ_dH.$$

Specially, when $D = I$, we get

$$M_d = A_d + KZ_dH.$$

Moreover, if Z is nonsingular, then

$$M_d = A_d + KZ^{-1}H.$$

From Corollary 2.1, when $C = I$, we get a result of perturbation of the Drazin inverse.

Corollary 2.2 ([10]). Suppose $B(I - AA_d) = 0,\ (I - AA_d)D_d = 0$ and $\|A_d\| \cdot \|D_dB\| \leq 1$, then

$$(A - D_dB)_d = (I - A_dD_dB)^{-1}A_d = A_d(I - D_dBA_d)^{-1}$$

and

$$(A - D_dB)_d - A_d = (A - D_dB)_dD_dBA_d = A_dD_dB(A - D_dB)_d,$$

with

$$\frac{\|(A - D_dB)_d - A_d\|}{\|A_d\|} \leq \frac{k_d(A)\|D_dB\|/\|A\|}{1 - k_d(A)\|D_dB\|/\|A\|},$$

where $k_d(A) = \|A\|\|A_d\|$ is the condition number with respect to the Drazin inverse.

Theorem 2.2. Suppose $P = 0,\ Q = 0,\ Z = 0,\ C(I - WD_{d,w}WD)W\Gamma_{d,w}WB = 0,\ CWD_{d,w}W(I - \Gamma W\Gamma_{d,w}W)B = 0,\ C(I - W\Gamma_{d,w}W\Gamma)WD_{d,w}WB = 0$ and $CW\Gamma_{d,w}W(I - DWD_{d,w}W)B = 0$, then

$$M_{d,w} = (I - KW\Gamma_{d,w}WHW)A_{d,w}(I - WKW\Gamma_{d,w}WH). \tag{2.6}$$

Proof. Let the right hand side of (2.6) be X. Firstly, we have

$$\begin{aligned} & MWX \\ = \ & (A - CWD_{d,w}WB)W(I - KW\Gamma_{d,w}WHW)A_{d,w}(I - WKW\Gamma_{d,w}WH) \\ = \ & (AW - CWD_{d,w}WBW)(A_{d,w} - A_{d,w}WKW\Gamma_{d,w}WH \\ & -KW\Gamma_{d,w}WHWA_{d,w} + KW\Gamma_{d,w}WHWA_{d,w}WKW\Gamma_{d,w}WH) \\ = \ & AWA_{d,w} - AWA_{d,w}WKW\Gamma_{d,w}WH - AWKW\Gamma_{d,w}WHWA_{d,w} \\ & +AWKW\Gamma_{d,w}WHWA_{d,w}WKW\Gamma_{d,w}WH - CWD_{d,w}WBWA_{d,w} \\ & +CWD_{d,w}WBWA_{d,w}WKW\Gamma_{d,w}WH + CWD_{d,w}WBWKW\Gamma_{d,w} \\ & \times WHWA_{d,w} - CWD_{d,w}WBWKW\Gamma_{d,w}WHWA_{d,w}WKW\Gamma_{d,w}WH \end{aligned}$$

$$
\begin{aligned}
&= AWA_{d,w} - KW\Gamma_{d,w}WH - CW\Gamma_{d,w}WHWA_{d,w} \\
&\quad +CW\Gamma_{d,w}WHWA_{d,w}WKW\Gamma_{d,w}WH - CWD_{d,w}WBWA_{d,w} \\
&\quad +CWD_{d,w}W\Gamma W\Gamma_{d,w}WH + CWD_{d,w}WDW\Gamma_{d,w}WHWA_{d,w} \\
&\quad -CWD_{d,w}WDW\Gamma_{d,w}WHWA_{d,w}WKW\Gamma_{d,w}WH \\
&= AWA_{d,w} - KW\Gamma_{d,w}WH
\end{aligned}
$$

and

$$
\begin{aligned}
&\quad XWM \\
&= (I - KW\Gamma_{d,w}WHW)A_{d,w}(I - WKW\Gamma_{d,w}WH)W(A - CWD_{d,w}WB) \\
&= (A_{d,w} - A_{d,w}WKW\Gamma_{d,w}WH - KW\Gamma_{d,w}WHWA_{d,w} \\
&\quad +KW\Gamma_{d,w}WHWA_{d,w}WKW\Gamma_{d,w}WH)(WA - WCWD_{d,w}WB) \\
&= A_{d,w}WA - A_{d,w}WCWD_{d,w}WB - A_{d,w}WKW\Gamma_{d,w}WHWA \\
&\quad +A_{d,w}WKW\Gamma_{d,w}WHWCWD_{d,w}WB - KW\Gamma_{d,w}WHWA_{d,w}WA \\
&\quad +KW\Gamma_{d,w}WHWA_{d,w}WCWD_{d,w}WB + KW\Gamma_{d,w}WHWA_{d,w}WKW\Gamma_{d,w} \\
&\quad \times WHWA - KW\Gamma_{d,w}WHWA_{d,w}WKW\Gamma_{d,w}WHWCWD_{d,w}WB \\
&= A_{d,w}WA - A_{d,w}WCWD_{d,w}WB - A_{d,w}WKW\Gamma_{d,w}WB \\
&\quad +A_{d,w}WKW\Gamma_{d,w}WDWD_{d,w}WB - KW\Gamma_{d,w}WH \\
&\quad +KW\Gamma_{d,w}W\Gamma WD_{d,w}WB + KW\Gamma_{d,w}WHWA_{d,w}WKW\Gamma_{d,w}WB \\
&\quad -KW\Gamma_{d,w}WHWA_{d,w}WKW\Gamma_{d,w}WDWD_{d,w}WB \\
&= A_{d,w}WA - KW\Gamma_{d,w}WH,
\end{aligned}
$$

i.e.,

$$MWX = XWM. \tag{2.7}$$

Secondly, we get

$$
\begin{aligned}
&\quad XWMWX \\
&= (A_{d,w}WA - KW\Gamma_{d,w}WH)W(I - KW\Gamma_{d,w}WHW)A_{d,w} \\
&\quad \times(I - WKW\Gamma_{d,w}WH) \\
&= (A_{d,w}WAW - A_{d,w}WAWKW\Gamma_{d,w}WHW - KW\Gamma_{d,w}WHW \\
&\quad +KW\Gamma_{d,w}WHWKW\Gamma_{d,w}WHW)A_{d,w}(I - WKW\Gamma_{d,w}WH) \\
&= (A_{d,w}WAW - KW\Gamma_{d,w}WBWA_{d,w}W)A_{d,w}(I - WKW\Gamma_{d,w}WH) \\
&= (I - KW\Gamma_{d,w}WBWA_{d,w}W)A_{d,w}WAWA_{d,w}(I - WKW\Gamma_{d,w}WH) \\
&= (I - KW\Gamma_{d,w}WHW)A_{d,w}(I - WKW\Gamma_{d,w}WH) \\
&= X.
\end{aligned}
$$

Finally, we shall prove that

$$(MW)^{k+1}XW = (MW)^k, \tag{2.8}$$

by induction on $k \geq l = Ind(AW)$. For the case $l = Ind(AW) = 1$, it is easy to see from $(AW)^2A_{d,w}W = AW$ that

$$(MW)^2XW \;=\; MWMWXW$$

$$
\begin{aligned}
&= (A - CWD_{d,w}WB)W(AWA_{d,w} - KW\Gamma_{d,w}WH)W \\
&= (AWAWA_{d,w} - AWKW\Gamma_{d,w}WH - CWD_{d,w}WBWAWA_{d,w} \\
&\quad +CWD_{d,w}WBWKW\Gamma_{d,w}WH)W \\
&= (AWAWA_{d,w} - CWD_{d,w}WBWAWA_{d,w})W \\
&= (A - CWD_{d,w}WB)W \\
&= MW.
\end{aligned}
$$

Generally, for $l = Ind(AW) > 1$, note the fact $(AW)^{l+1}A_{d,w}W = (AW)^l$ that

$$
\begin{aligned}
&(MW)^{l+1}XW \\
&= (MW)^l MWXW \\
&= [(A - CWD_{d,w}WB)W]^l(AWA_{d,w} - KW\Gamma_{d,w}WH)W \\
&= [(A - CWD_{d,w}WB)W]^l AWA_{d,w}W(I - A_{d,w}WCW\Gamma_{d,w}WHW) \\
&= [(A - CWD_{d,w}WB)W]^l(I - A_{d,w}WCW\Gamma_{d,w}WHW) \\
&= [(A - CWD_{d,w}WB)W]^l - [(A - CWD_{d,w}WB)W]^l A_{d,w}WCW\Gamma_{d,w}WHW \\
&= [(A - CWD_{d,w}WB)W]^l - [(A - CWD_{d,w}WB)W]^{l-1} \\
&\quad \times(AWA_{d,w}WCW\Gamma_{d,w}WHW - CWD_{d,w}WBWA_{d,w}WCW\Gamma_{d,w}WHW) \\
&= [(A - CWD_{d,w}WB)W]^l - [(A - CWD_{d,w}WB)W]^{l-1} \\
&\quad \times(CW\Gamma_{d,w}WHW - CWD_{d,w}WDW\Gamma_{d,w}WHW) \\
&= [(A - CWD_{d,w}WB)W]^l \\
&= (MW)^l.
\end{aligned}
$$

For $k \geq l = Ind(AW)$, it is easy to verify that

$$(MW)^{k+1}XW = (MW)^k.$$

Therefore, (2.8) holds, which completes the proof. □

By Theorem 2.2, when $A,\ B,\ C,\ D$ are square and $W = I$, we can directly get Theorem 2.2 in [17].

Corollary 2.3 ([17]). Suppose $P = 0,\ Q = 0,\ Z = 0,\ C(I - DD_d)\Gamma_d B = 0,\ CD_d(I - \Gamma\Gamma_d)B = 0,$ $C(I - \Gamma\Gamma_d)D_dB = 0$ and $C\Gamma_d(I - DD_d)B = 0$, then

$$M_d = (I - K\Gamma_d H)A_d(I - K\Gamma_d H).$$

By Corollary 2.3, when $D = I$, we get Theorem 2.2 in [11].

Corollary 2.4 ([11]). Suppose $P = 0,\ Q = 0,\ Z = 0$, and $C(I - \Gamma\Gamma_d)B = 0$, then

$$M_d = (A - CB)_d = (I - K\Gamma_d H)A_d(I - K\Gamma_d H).$$

Next, we present another result of this paper.

Theorem 2.3. Suppose $P = 0,\ Q = 0,\ Ind(ZW) = 1,\ C(I - WD_{d,w}WD) = 0,\ (I - DWD_{d,w}W)B = 0,\ CWD_{d,w}W(I - \Gamma W\Gamma_{d,w}W) = 0,\ (I - W\Gamma_{d,w}W\Gamma)WD_{d,w}WB = 0$ and $WZ_{d,w}WZW\Gamma_{d,w}W = W\Gamma_{d,w}WZWZ_{d,w}W$, then

$$M_{d,w} = [I - KW(I - Z_{d,w}WZW)\Gamma_{d,w}WHW]A_{d,w} \times$$

$$[I - WKW\Gamma_{d,w}W(I - ZWZ_{d,w}W)H] + KWZ_{d,w}WH. \tag{2.9}$$

Proof. Let the right hand side of (2.9) be X. First, note the facts:

$$\begin{aligned}
& MW[I - KW(I - Z_{d,w}WZW)\Gamma_{d,w}WHW] \\
= \ & MW - (AW - CWD_{d,w}WBW)(KW\Gamma_{d,w}WHW \\
& -KWZ_{d,w}WZW\Gamma_{d,w}WHW) \\
= \ & MW - AWKW\Gamma_{d,w}WHW + AWKWZ_{d,w}WZW\Gamma_{d,w}WHW \\
& +CWD_{d,w}WBWKW\Gamma_{d,w}WHW \\
& -CWD_{d,w}WBWKWZ_{d,w}WZW\Gamma_{d,w}WHW \\
= \ & MW
\end{aligned}$$

similarly, we get

$$[I - WKW\Gamma_{d,w}W(I - ZWZ_{d,w}W)H]WM = WM,$$

now, we have

$$\begin{aligned}
& MWX \\
= \ & MW[I - KW(I - Z_{d,w}WZW)\Gamma_{d,w}WHW]A_{d,w} \\
& \times[I - WKW\Gamma_{d,w}W(I - ZWZ_{d,w}W)H] + MWKWZ_{d,w}WH \\
= \ & (AW - CWD_{d,w}WBW)A_{d,w}[I - WKW\Gamma_{d,w}W(I - ZWZ_{d,w}W)H] \\
& +(AW - CWD_{d,w}WBW)KWZ_{d,w}WH \\
= \ & AWA_{d,w} - AWA_{d,w}WKW\Gamma_{d,w}W(I - ZWZ_{d,w}W)H \\
& -CWD_{d,w}WBWA_{d,w} + CWD_{d,w}WBWA_{d,w}WKW\Gamma_{d,w}W(I \\
& -ZWZ_{d,w}W)H + AWKWZ_{d,w}WH - CWD_{d,w}WBWKWZ_{d,w}WH \\
= \ & AWA_{d,w} - KW\Gamma_{d,w}WH + KW\Gamma_{d,w}WZWZ_{d,w}WH - CWD_{d,w}WH \\
& +CWD_{d,w}W\Gamma W\Gamma_{d,w}WH - CWD_{d,w}W\Gamma W\Gamma_{d,w}WZWZ_{d,w}WH \\
& +CWZ_{d,w}WH - CWD_{d,w}WDZ_{d,W}H + CWD_{d,w}WZWZ_{d,w}WH \\
= \ & AWA_{d,w} - KW\Gamma_{d,w}WH + KW\Gamma_{d,w}WZWZ_{d,w}WH - CWD_{d,w}W \\
& \times(I - \Gamma W\Gamma_{d,w}W)H + CWD_{d,w}W(I - \Gamma W\Gamma_{d,w}W)ZWZ_{d,w}WH \\
= \ & AWA_{d,w} - KW\Gamma_{d,w}WH + KW\Gamma_{d,w}WZWZ_{d,w}WH
\end{aligned}$$

and

$$\begin{aligned}
& XWM \\
= \ & [I - KW(I - Z_{d,w}WZW)\Gamma_{d,w}WHW]A_{d,w} \\
& \times[I - WKW\Gamma_{d,w}W(I - ZWZ_{d,w}W)H]WM + KWZ_{d,w}WHWM \\
= \ & [I - KW(I - Z_{d,w}WZW)\Gamma_{d,w}WHW]A_{d,w}(WA - WCWD_{d,w}WB) \\
& +KWZ_{d,w}WH(WA - WCWD_{d,w}WB) \\
= \ & A_{d,w}WA - KW(I - Z_{d,w}WZW)\Gamma_{d,w}WHWA_{d,w}WA \\
& -A_{d,w}WCWD_{d,w}WB + KW(I - Z_{d,w}WZW)\Gamma_{d,w}WHWA_{d,w}W \\
& \times CWD_{d,w}WB + KWZ_{d,w}WHWA - KWZ_{d,w}WHWCWD_{d,w}WB \\
= \ & A_{d,w}WA - KW\Gamma_{d,w}WH + KWZ_{d,w}WZW\Gamma_{d,w}WH - KWD_{d,w}WB \\
& +KW\Gamma_{d,w}W\Gamma WD_{d,w}WB - KWZ_{d,w}WZW\Gamma_{d,w}W\Gamma WD_{d,w}WB
\end{aligned}$$

$$\begin{aligned}
&+KWZ_{d,w}WB - KWZ_{d,w}WDWD_{d,w}WB + KWZ_{d,w}WZWD_{d,w}WB \\
=\ & A_{d,w}WA - KW\Gamma_{d,w}WH + KWZ_{d,w}WZW\Gamma_{d,w}WH - K(I - W\Gamma_{d,w}W\Gamma) \\
&\times WD_{d,w}WB + KWZ_{d,w}WZW(I - W\Gamma_{d,w}W\Gamma)WD_{d,w}WB \\
=\ & A_{d,w}WA - KW\Gamma_{d,w}WH + KWZ_{d,w}WZW\Gamma_{d,w}WH,
\end{aligned}$$

i.e.,

$$MWX = XWM. \tag{2.10}$$

Secondly, we get

$$\begin{aligned}
XWMWX =\ & (A_{d,w}WA - KW\Gamma_{d,w}WH + KWZ_{d,w}WZW\Gamma_{d,w}WH)W \\
&\times[I - KW(I - Z_{d,w}WZW)\Gamma_{d,w}WHW]A_{d,w} \\
&\times[I - WKW\Gamma_{d,w}W(I - ZWZ_{d,w}W)H] + (A_{d,w}WA \\
&-KW\Gamma_{d,w}WH + KWZ_{d,w}WZW\Gamma_{d,w}WH)WKWZ_{d,w}WH \\
=\ & (I - KW\Gamma_{d,w}WHW + KWZ_{d,w}WZW\Gamma_{d,w}WHW) \\
&\times[I - KW(I - Z_{d,w}WZW)\Gamma_{d,w}WHW]A_{d,w}WAWA_{d,w} \\
&\times[I - WKW\Gamma_{d,w}W(I - ZWZ_{d,w}W)H] + KWZ_{d,w}WH \\
=\ & (I - KW\Gamma_{d,w}WHW + KWZ_{d,w}WZW\Gamma_{d,w}WHW) \\
&\times(I - KW\Gamma_{d,w}WHW + KWZ_{d,w}WZW\Gamma_{d,w}WHW)A_{d,w} \\
&\times[I - WKW\Gamma_{d,w}W(I - ZWZ_{d,w}W)H] + KWZ_{d,w}WH \\
=\ & [I - KW(I - Z_{d,w}WZW)\Gamma_{d,w}WHW]A_{d,w} \\
&\times[I - WKW\Gamma_{d,w}W(I - ZWZ_{d,w}W)H] + KWZ_{d,w}WH \\
=\ & X.
\end{aligned}$$

Finally, we shall prove that

$$(MW)^{k+1}XW = (MW)^k, \tag{2.11}$$

by induction on $k \geq l = Ind(AW)$. For $l = Ind(AW)$, note the facts:

$$MW(I - KW(I - Z_{d,w}WZW)\Gamma_{d,w}WHW) = MW$$

and

$$(MW)^l AWA_{d,w}W = (MW)^l.$$

Now, we have

$$\begin{aligned}
&(MW)^{l+1}XW \\
=\ & (MW)^l MWXW \\
=\ & (MW)^l(AWA_{d,w}W - KW\Gamma_{d,w}WHW + KW\Gamma_{d,w}WZWZ_{d,w}WHW) \\
=\ & (MW)^l(I - KW\Gamma_{d,w}WH + KW\Gamma_{d,w}WZWZ_{d,w}WH)WAWA_{d,w}W \\
=\ & (MW)^{l-1}[MW(I - KW\Gamma_{d,w}WH + KW\Gamma_{d,w}WZWZ_{d,w}WH)]WAWA_{d,w}W \\
=\ & (MW)^{l-1}MWAWA_{d,w}W \\
=\ & (MW)^l.
\end{aligned} \tag{2.12}$$

For $k \geq l = Ind(AW)$. From (2.12), we get (2.11), which completes the proof. □

When $A,\ B,\ C,\ D$ are square and $W = I$, we get the following corollary.

Corollary 2.5. Suppose $P = 0,\ Q = 0,\ Ind(Z) = 1,\ C(I-DD_d) = 0,\ (I-DD_d)B = 0,\ CD_d(I-\Gamma\Gamma_d) = 0,$ $(I-\Gamma_d\Gamma)D_dB = 0$ and $Z_dZ\Gamma_d = \Gamma_dZZ_d$, then

$$M_d = [I - K(I - ZZ_d)\Gamma_dH]A_d[I - K\Gamma_d(I - ZZ_d)H] + KZ_dH.$$

By Corollary 2.5, when $D = I$, we have the following result.

Corollary 2.6. Suppose $P = 0,\ Q = 0,\ Ind(Z) = 1,\ C(I-\Gamma\Gamma_d) = 0,\ (I-\Gamma_d\Gamma)B = 0$ and $Z_dZ\Gamma_d = \Gamma_dZZ_d$, then

$$M_d = (A - CB)_d = [I - K(I - ZZ_d)\Gamma_dH]A_d[I - K\Gamma_d(I - ZZ_d)H] + KZ_dH.$$

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