KALAHARI JOURNALS

Inverse Heat Conduction Problem in a Semi-infinite Hollow Cylinder and its Thermal Deflection by Quasi-static Approach

G.D. Kedar(✉)[a, *], K.C. Deshmukh[a, †]

[a]Department of mathematics, RTM Nagpur University, Nagpur-440033 (Maharashtra) India.

ABSTRACT

In this study inverse heat conduction problem is to simultaneously determine unknown temperature and thermal deflection on the outer curved surface of a semi-infinite hollow circular cylinder from the knowledge of temperature distribution within the cylinder. The hollow circular cylinder is subjected to an arbitrary known temperature under unsteady state condition. Initially the cylinder is at zero temperature and temperature at the lower surface is at zero heat flux. Also the inner boundary surface of the cylinder is at zero temperature. The governing heat conduction equation has been solved by using integral transform method. The results are obtained in series form in terms of Bessels functions. Mathematical model has also been constructed with the help of numerical illustration.

1. Introduction

The inverse thermoelastic problem consists of determination of the temperature of the heating medium, the heat flux on the boundary surfaces of the solid when the conditions of the displacement and stresses are known at the some points of the hollow cylinder under consideration. Conventionally these quantities were obtained by actual experiments. The experimental methods require equipments and manual efforts as well as it may also consume large time. While the studies of inverse heat conduction problems provides best alternative to the experimental methods due to which one can avoid all constraints of it and can determine these quantities quite easily and with great accuracy. Now in recent years many analytical and numerical techniques have been developed to solve inverse heat conduction problems.

Sabherwal [1, 2] studied inverse problem in heat conduction. Deshmukh and Wankhede [3] solved an inverse problem of thermoelasticity in a thin circular plate by determining the temperature on the curved surface of the plate, displacement and thermal stresses using quasi-static approach by employing integral transform techniques. Khobragade and Deshmukh [4], studied an inverse axially symmetric quasi-static problem of thermoelasticity for a thin clamped circular plate in which a heat flux is prescribed on an internal cylindrical surface of the plate and suitable heat exchange conditions are met on the upper and lower surfaces of the plate is solved with the help of a generalized integral transform technique. Tikhe and Deshmukh [5] studied the inverse heat conduction problem in a thin circular plate and its thermal deflection on the outer curved surface. Kulkarni and Deshmukh [6], studied an inverse quasi-static steady state thermal stresses in a thick circular plate. Recently Deshmukh et. al [7], studied an quasi-static thermal deflection of a thin clamped circular plate due to heat generation. Very recently Deshmukh et. al [8] studied inverse heat conduction problem for a semi-infinite circular plate and discussed thermal deflection.

✉ Corresponding author.
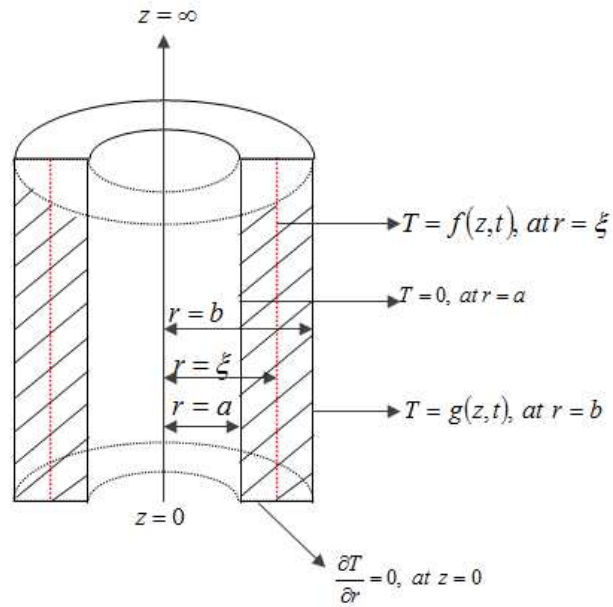*gdkedar@rediffmail.co.in
† kcdeshmukh2000@rediffmail.com

Figure 1: Geometry of the Problem

In this paper the hollow cylinder is subjected to an arbitrary known temperature under unsteady state condition considered and discussed the thermal deflection where the inner surface is built in. Initially the cylinder is at zero temperature and temperature at the lower surface is at zero heat flux. Also the inner boundary surface of the cylinder $r = a$ is at zero temperature. The governing heat conduction equation has been solved by using integral transform method. The results are obtained in series form in terms of Bessel's functions. Mathematical model has also been constructed with the help of numerical illustration. No one previously studied such type of problem. This is a new contribution to the field.

The inverse problem is very important in view of its relevance to various industrial mechanics subjected to heating such as the main shaft of lathe, turbines, and the role of the rolling mill. Also arise the quenching studies, the analysis of experimental data and measurement of aerodynamic heating. Also typical practical applications include determination of the temperature and the heat flux at the highly heated outer surface of a recently vehicle in the atmosphere from measurements taken inside the body, calorimeter type instrumentation, combustion chamber etc.

2. Formulation of the problem

Consider a hollow cylinder defined by $a \leq r \leq b$, $0 \leq z < \infty$. Let the cylinder be subjected to arbitrary known interior temperature $f(z,t)$ within the region, $a \leq r \leq b$. Initially the cylinder is at zero temperature and temperature at the lower surface $z = 0$ is at zero heat flux. Also the inner boundary surface of the cylinder $r = a$ is at zero temperature. Under these more realistic prescribed conditions, the unknown temperature $g(z,t)$ on the outer surface of hollow cylinder at $r = b$ and quasi-static thermal deflection due to unknown temperature $g(z,t)$ are required to be determined. The differential equation satisfying the deflection function $W(r,t)$ is given as

$$\nabla^4 W = -\frac{\nabla^2 M_T}{D(1-\nu)} \tag{2.1}$$

Where, M_T is the thermal moment of the hollow cylinder defined as

$$M_T = a_t \, E \int_0^\infty T\,(r,z,t)\; z\,dz \qquad (2.2)$$

D is the flexural rigidity of the hollow cylinder denoted as

$$D = \frac{Eh^3}{12\,(1 - \nu^2)} \qquad (2.3)$$

a_t, E and ν are the coefficients of the linear thermal expansion, the Young's modulus and Poisson's ratio of the hollow cylinder material respectively and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \qquad (2.4)$$

Since, the edge of the hollow cylinder is fixed and clamped;

$$W = \frac{\partial W}{\partial r} = 0 \text{ at } r = a \qquad (2.5)$$

$$\text{Initially } T = W = 0 \text{ when } t = 0 \qquad (2.6)$$

The temperature of the hollow cylinder satisfies the heat conduction equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \;\; a \le r \le b,\, 0 \le z < \infty, \;\; t > 0 \qquad (2.7)$$

with the boundary conditions

$$T(r,z,t) = 0 \text{ at } r = a \qquad (2.8)$$

$$T(b,z,t) = g(z,t)\,(\text{Unknown}) \qquad (2.9)$$

$$\frac{\partial T}{\partial z} = 0 \;\text{ at } z = 0 \qquad (2.10)$$

$$T(r,z,t) = 0 \text{ at } z = \infty \qquad (2.11)$$

and interior condition

$$T(\xi,z,t) = f(z,t)\;(\text{Known}) \qquad (2.12)$$

and the initial condition

$$T(r,z,t) = 0 \;\text{ when } t = 0 \qquad (2.13)$$

where, α is thermal diffusivity of the material of the cylinder.

3. Solution of the problem

The equations 2.7 - 2.13 defines boundary value problem of heat conduction. This problem is solved by using technique of integral transform suggested by Ozisik [9]. Applying Fourier cosine transform over the variable z and then applying Laplace transform one obtains the reduced system as

$$\frac{\partial^2 \overline{T}^*}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{T}^*}{\partial r} - q^2 \overline{T}^* = 0 \tag{3.1}$$

with

$$\overline{T}^*(a, \beta, s) = 0 \tag{3.2}$$

$$\overline{T}^*(b, \beta, s) = \overline{g}^*(\beta, s) \tag{3.3}$$

$$\overline{T}^*(\xi, \beta, s) = \overline{f}^*(\beta, s) \tag{3.4}$$

where, $q^2 = \beta^2 + \frac{s}{\alpha}$

where, $\overline{T}(r, \beta, t)$ is the infinite cosine transform of $T(r, z, t), \beta$ is the cosine transform parameter, $\overline{T}^*$ denotes Laplace transform of $\overline{T}$ and s is the Laplace transform parameter.

The solution of equation 3.1 is obtain in terms of modified Bessel's functions of first and second kind $I_0(qr)$ and $K_0(qr)$ of order zero respectively. Then applying conditions 3.2 – 3.4 one obtains solution as

$$\overline{T}^*(r, \beta, s) = -\overline{f}^*(\beta, s) \left[\frac{I_0(qr)\, K_0(qa) - I_0(qa)\, K_0(qr)}{I_0(qa)\, K_0(q\xi) - I_0(q\xi)\, K_0(qa)} \right] \tag{3.5}$$

and

$$\overline{g}^*(\beta, s) = -\overline{f}^*(\beta, s) \left[\frac{I_0(qb)\, K_0(qa) - I_0(qa)\, K_0(qb)}{I_0(qa)\, K_0(q\xi) - I_0(q\xi)\, K_0(qa)} \right] \tag{3.6}$$

Finally employing the Laplace inverse transform and inverse cosine transform the temperature distribution obtained as

$$
\begin{aligned}
T(r,z,t) = &-2\alpha\sqrt{\tfrac{2}{\pi}} \int_{\beta=0}^{\infty} Cos\,(\beta\, z) \times \sum_{n=1}^{\infty} \left\{ \frac{\lambda_n^2\, J_0(\lambda_n a)\, J_0(\lambda_n \xi)}{[J_0^2(\lambda_n \xi) - J_0^2(\lambda_n a)]} \right. \\
&\left. \times\, [J_0(\lambda_n r)\, Y_0(\lambda_n a) - J_0(\lambda_n a)\, Y_0(\lambda_n r)] \times \left[\int_0^{t} \overline{f}(\beta, t')\, e^{-\alpha\left(\lambda_n^2 + \beta^2\right)\,(t-t')} dt' \right] \right\} d\beta
\end{aligned}
\tag{3.7}
$$

and the expression for $g(z, t)$, the unknown temperature at $r = b$ as

$$
\begin{aligned}
g(z,t) = &-2\alpha\sqrt{\tfrac{2}{\pi}} \int_{\beta=0}^{\infty} Cos\,(\beta\, z) \times \sum_{n=1}^{\infty} \left\{ \frac{\lambda_n^2\, J_0(\lambda_n a)\, J_0(\lambda_n \xi)}{[J_0^2(\lambda_n \xi) - J_0^2(\lambda_n a)]} \right. \\
&\left. \times\, [J_0(\lambda_n b)\, Y_0(\lambda_n a) - J_0(\lambda_n a)\, Y_0(\lambda_n b)] \times \left[\int_0^{t} \overline{f}(\beta, t')\, e^{-\alpha\left(\lambda_n^2 + \beta^2\right)\,(t-t')} dt' \right] \right\} d\beta
\end{aligned}
\tag{3.8}
$$

where, λ_n are positive roots of the transcendental equation

$$J_0(\lambda_n a)\, Y_0(\lambda_n \xi) - J_0(\lambda_n \xi)\, Y_0(\lambda_n a) = 0 \tag{3.9}$$

Using equation 3.7 into equation 2.2, one obtains

$$
M_T = -2\alpha\sqrt{\tfrac{2}{\pi}}\, a_t\, E \int_0^\infty z \left[\int_{\beta=0}^\infty Cos\,(\beta z) \times \sum_{n=1}^\infty \left\{ \frac{\lambda_n^2\, J_0(\lambda_n a)\, J_0(\lambda_n \xi)}{\left[J_0^2(\lambda_n \xi) - J_0^2(\lambda_n a)\right]} \right. \right.
$$
$$
\left. \left. \times\, [J_0(\lambda_n r)\, Y_0(\lambda_n a) - J_0(\lambda_n a)\, Y_0(\lambda_n r)] \times \left[\int_0^t \overline{f}(\beta, t')\, e^{-\alpha\left(\lambda_n^2 + \beta^2\right)\,(t-t')}dt' \right] \right\} d\beta \right] dz \tag{3.10}
$$

Hence,

$$
\nabla^2 M_T = \left(2\alpha\sqrt{\tfrac{2}{\pi}}\, a_t\, E\right) \int_0^\infty z \left[\int_{\beta=0}^\infty Cos\,(\beta z) \times \sum_{n=1}^\infty \left\{ \frac{\lambda_n^4\, J_0(\lambda_n a)\, J_0(\lambda_n \xi)}{\left[J_0^2(\lambda_n \xi) - J_0^2(\lambda_n a)\right]} \right. \right.
$$
$$
\left. \left. \times\, [J_0(\lambda_n r)\, Y_0(\lambda_n a) - J_0(\lambda_n a)\, Y_0(\lambda_n r)] \times \left[\int_0^t \overline{f}(\beta, t')\, e^{-\alpha\left(\lambda_n^2 + \beta^2\right)\,(t-t')}dt' \right] \right\} d\beta \right] dz \tag{3.11}
$$

Assume the solution of the equation 2.1 satisfy condition 2.5 as

$$
W(r,t) = \sum_{n=1}^\infty C_n(t)\left\{ 2a\, [J_0(\lambda_n r)\, Y_0\,(\lambda_n a) - J_0(\lambda_n a)\, Y_0\,(\lambda_n r)] \right.
$$
$$
\left. +\left(r^2 - a^2\right)\lambda_n\, [J_1(\lambda_n a)\, Y_0(\lambda_n a) - J_0(\lambda_n a)\, Y_1\,(\lambda_n a)] \right\} \tag{3.12}
$$

Substituting equation 3.11 and 3.12 into equation (2.1), one obtains the expression for thermal deflection as

$$
\frac{W(r,t)}{P} = \sum_{n=1}^\infty \frac{J_0(\lambda_n a)\, J_0(\lambda_n \xi)}{\left[J_0^2(\lambda_n \xi) - J_0^2(\lambda_n a)\right]} \times \left\{ 2a\, [J_0(\lambda_n r)\, Y_0\,(\lambda_n a) - J_0(\lambda_n a)\, Y_0\,(\lambda_n r)] \right.
$$
$$
\left. +\left(r^2 - a^2\right)\lambda_n\, [J_1(\lambda_n a)\, Y_0(\lambda_n a) - J_0(\lambda_n a)\, Y_1\,(\lambda_n a)] \right\}
$$
$$
\times \left(\int_{z=0}^\infty z \left\{ \int_{\beta=0}^\infty Cos\,(\beta z) \left[\int_{t'=0}^t \overline{f}(\beta, t')\, e^{-\alpha\left(\lambda_n^2 + \beta^2\right)\,(t-t')}dt' \right] d\beta \right\} dz \right) \tag{3.13}
$$

4. Numerical calculation

In the previous sections one has formulated and solved the problem of heat conduction and thermoelasticity. For the numerical calculation one consider a special case of copper hollow cylinder with specifications as defined below.

Special Function
$f(z,t) = (1 - e^{-\omega t})\, z^2\, e^{-z}$ with $\omega > 0$, $t \to t' = 5$ sec.
Dimensions of cylider
Inner radius $a = 1\ m$
Outer radius $b = 2\ m$
Material properties
Thermal diffusivity $\alpha = 112.34 \times 10^{-6}\ m^2 s^{-1}$
Density $\rho = 8954\ kg\, m^{-3}$
Specific heat $c_p = 383\ J\, kg^{-1} K^{-1}$
Poisson ratio $\nu = 0.35$
Coefficient of linear thermal expansion, $a_t = 16.5 \times 10^{-6}\ K^{-1}$
Lame$'$ constant $\mu = 26.67$.
For convenience one sets
$P = \sqrt{\tfrac{2}{\pi}}\, \frac{\alpha\, a_t\, E}{a\, D(1-\nu)}\, 10^3$, $X_1 = 2\alpha\,\sqrt{\tfrac{2}{\pi}}\, 10^4$ and $X_2 = 2\alpha\,\sqrt{\tfrac{2}{\pi}}\, 10^6$.
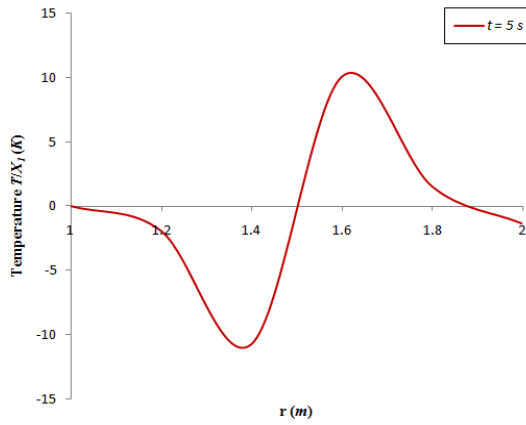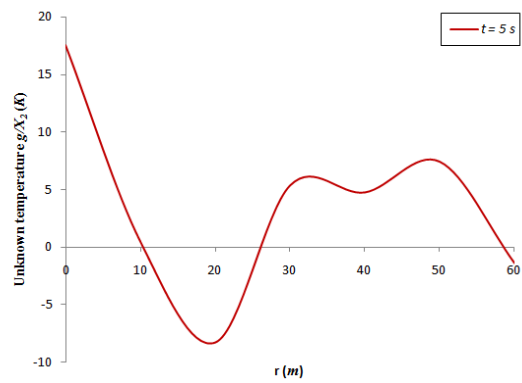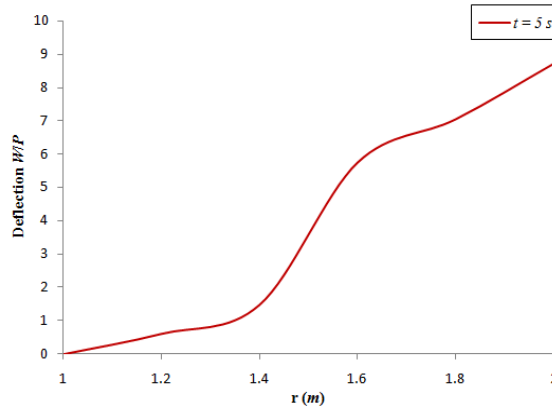
Figure 2: Temperature distribution along radial direcion at $z = 10m$.

Figure 3: Unknown Temperature distribution along axial direction at $r = 2m$.

Figure 4: Thermal deflection along radial direction.

Fig. 2 shows the temperature distribution along radial direction at $z = 10m$. Due to prescribed interior heat source the temperature changes its profile at $r = 1.5m$ as the direction of heat flow is in opposite. Temperature decreases in the region $1 \leq r \leq 1.4m$ and then suddenly increases in the region $1.4 \leq r \leq 1.6m$. In the region $1.6 \leq r \leq 2m$ it remains steady and negligible towards outer curved surface $r = 2m$ of the hollow cylinder. In Fig. 3 it is observed that unknown temperature distribution at outer curved surface $r = 2m$ along axial direction decreases in the annular region $0 \leq z \leq 20m$. Temperature rapidly increases in region $20 \leq z \leq 30m$ and then steady towards upper end. The maximum temperature is observed at lower edge $z = 0$. In Fig. 4, thermal deflection goes on increasing from inner to outer curved surface of the semi-infinite hollow cylinder. Since, the inner curved surface is fixed and clamped, the thermal deflection is zero at this surface and also negligible in the region $0 \leq r \leq 1.2m$. Thermal deflection increases rapidly in the region $1.4 \leq r \leq 1.6m$.

5. Conclusions

In this problem one solved the inverse problem of thermoelasticity and determined the unknown temperature on the outer curved surface and thermal deflection in a semi-infinite hollow cylinder. As a special case, Mathematical model is constructed for Copper semi-infinite solid cylinder with the material properties specified in the numerical

calculations.

Here, one consider a semi-infinite hollow cylinder subjected to interior temperature at the region $r = \xi = 1.5m$, $0 \leq z < \infty$ in the form of $f(z,t) = (1 - e^{-\omega t}) z^2 e^{-z}$. Due to the prescribed interior heat source, the temperature distribution observed at level of height $z = 10m$. The temperature changes its profile at $r = 1.5m$ as the direction of heat flow is in opposite and remains high in the region 0.1 neighborhood of $r = \xi$. Unknown temperature distribution is observed at outer curved surface which decreases from lower end to upper end of the cylinder.

Due to built-in inner curved boundary surface of the cylinder, the thermal deflection increases from inner to outer curved surface of the cylinder. Due to the prescribed interior heat source at $r = 1.5m$, the thermal deflection increases rapidly in the region $1.4 \leq r \leq 1.6m$ and maximum deflection occurs at outer curved surface.

Acknowledgements

References

[1] Sabherwal K.C., 1965. Indian Journal of Pure and Applied Physics, 3(10), 397-398.

[2] Sabherwal K.C., 1966. Indian Journal of Pure and Applied Physics, vol. XL(8), 448-450.

[3] Deshmukh K.C., Wankhede P.C., 1998. An inverse quasi-static transient thermoelastic problem in a thin circular plate, Bulletin of Pure and Applied Sciences, 17(1), 209-207.

[4] Khobragade N.L., Deshmukh K.C., 2005. An inverse quasi-static thermal deflection problem for a thin clamped circular plate, J. Thermal Stresses, 28(4), 353-361.

[5] Tikhe A.K. and Deshmukh K. C., 2006. Inverse heat conduction problem in a thin circular plate and its thermal deflection, Journal of Applied Mathematical Modelling, 30, 554-560.

[6] Kulkarni V.S., Deshmukh K.C., 2008. An inverse quasi-static thermal stresses in a thick circular plate J. Franklin Institute, 345, 29-38.

[7] Deshmukh K.C., Warbhe S. D., Kulkarni V.S., 2009. Quasi-static thermal deflection of a thin clamped circular plate due to heat generation, Journal of Thermal Stresses, 32(9), 877-886.

[8] Deshmukh K.C., Warbhe S.D., Kedar G.D. and Kulkarni V.S., 2010. Inverse Heat Conduction Problem in a Semi-infinite Circular Plate and its Thermal Deflection by Quasi-static Approach, Applications and Applied Mathematics: An International Journal (AAM), 5(1), 120-127.

[9] Ozisik M.N., 1968. Boundary value problems of heat conduction, International Text Book Company, Scranton, Pennsylvania.