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The Square Free Detour Number of a Graph

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Abstract **- For a connected graph $G = (V, E)$, the square free detour distance $D_{\square f}(u, v)$ is the length of a longest $u - v$ square free path in G, where u and v are the vertices of G. A $u - v$ square free path of length $D_{\square f}(u, v)$ is called the $u - v$ square free detour. In this article, we investigate the results on the square free detour number of standard graphs and special graphs. The relationship between the detour number and the square free detour number is exhibited. We also show that for any three integers α, β and γ with $3 \leq \alpha \leq \beta \leq \gamma$, there exists a connected graph G such that $dn(G) = \alpha$, $dn_{\square f}(G) = \beta$ and $dm(G) = \gamma$.**

Keywords - Detour number, detour monophonic number, Square free detour basis, Square free detour number.

1. INTRODUCTION

In a graph $G = (V, E)$ of order n, the distance $D(u, v)$ is known as the length of the longest $u - v$ path for vertices u and v of G. It is said to be an $u - v$ detour and denoted by $D(u, v)$. A vertex $w \in V(G)$ lies in an $u - v$ path P if w is an internal vertex of P distinct from u and v. For any vertex u of G, the detour eccentricity of u is $e_D(u) = max\{D(u, v): v \in V\}$. If $e_D(u) = D(u, v)$, then v is an eccentric vertex of u. The detour diameter and detour radius are denoted and defined as $diam_D G = max\{e_D(u): u \in V\}$ and $rad_D G = min\{e_D(u): u \in V\}$.

A vertex set S in G is called a detour set if each vertex of G lies in an $u - v$ detour such that $u, v \in S$. The minimum cardinality of a detour set S is said to be the detour number $dn(G)$. Any detour set with cardinality $dn(G)$ is said to be a detour basis. The geodetic number $g(G)$ based on the shortest path and the detour number $dn(G)$ based on the longest path were defined by Chartrand et al.[1]-[5] and developed by number of authors [10]-[12]. The concept of the detour monophonic number $dm(G)$, based on the detour chordless path was introduced by Titus et. al. [8].

The new concept of triangle free detour number and its parameters were studied by Sethu Ramalingam and Athisayanathan [7], [9]. The analogous concept of square free detour distance was studied by Priscilla Pacifica [6]. In this study, we introduce the concept based on square free detour distance called square free detour number of graphs and investigate the square free detour number for some classes of graphs.

In a connected graph G a path P is called a $u - v$ square free path if no four vertices of P induce a square where $u, v \in G$. The square free detour distance $D_{\square f}(u, v)$ is the length of the longest $u - v$ square free path in G. An $u - v$ path of length $D_{\square f}(u, v)$ is called an $u - v$ square free detour.

Throughout this article we consider a graph G to be a non-trivial, finite, simple and connected graph of order n.

2. SQUARE FREE DETOUR NUMBER OF A GRAPH

A set $S \subseteq V(G)$ is said to be a square free detour set of a connected graph G if every vertex of G lies on a square free detour joining a pair of vertices of S. The minimum cardinality of square free detour sets of G is called the square free detour number $dn_{\square f}(G)$ of G. A square free detour set of cardinality $dn_{\square f}(G)$ is a square free detour basis of G.

Example 2.2 We picture a graph G_1 in Figure 1, in which $S_1 = \{v_1, v_4\}$ of V is a square free detour basis of G_1 and so $dn_{\square f}(G_1) = 2$. The sets $S_2 = \{v_2, v_6\}$, $S_3 = \{v_3, v_4\}$, $S_4 = \{v_2, v_5\}$, $S_5 = \{v_1, v_6\}$ and $S_6 = \{v_3, v_5\}$ are also the square free detour bases for the graph G_1. Hence we notice that there can be many square free detour bases for a graph. Moreover, detour number and square free detour number are same for a graph depicted in Figure 1.

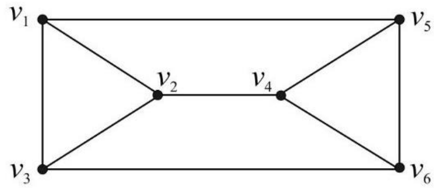

FIGURE 1: G_1

Remark 2.3 The detour number, the detour monophonic number and the square free detour number of the graph G are different. For the graph G_2 given in Figure 2, $S_1 = \{a, c\}$ is a detour basis of $G_2, S_2 = \{a, g, h\}$ is a square free detour basis and $S_3 = \{a, d, e, h\}$ is a detour monophonic basis of G_2. Therefore, $dn(G_1) = 2$, $dm(G_2) = 4$ and $dn_{\Box f}(G_2) = 3$.

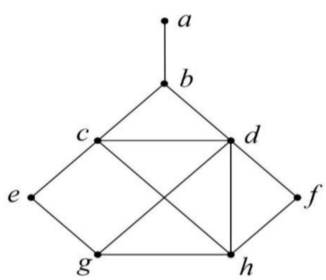

FIGURE 2: G_2

Theorem 2.4 For a connected graph G of order $n, 2 \leq dn_{\Box f}(G) \leq n$.

Proof. Since a square free detour set requires minimum of two vertices, $dn_{\Box f}(G) \geq 2$. Moreover, $V(G)$ is a square free detour set for a graph G and so $dn_{\Box f}(G) \leq n$. Hence $2 \leq dn_{\Box f}(G) \leq n$.

Remark 2.5 For any complete graph K_n, $dn_{\Box f}(G) = 2$ and for the path of order $2, dn_{\Box f}(G) = n$. Hence the bounds given in Theorem 2.4 hold sharp. Also, the bounds given in Theorem 2.4 hold strict for the graph depicted in Figure 1.

Theorem 2.6 Let G be a non-trivial graph. Then

(i) Every end-vertex of a non-trivial graph G belongs to every square free detour set of G.

(ii) If the set of all end-vertices of G is a square free detour set, then it is the unique square free detour basis of G.

(i) **Proof.** Let S be a square free detour set of G. Let v be an end-vertex of G. Assume $v \notin S$. Then v is an internal vertex in an $x - y$ square free detour path for some $x, y \in S$. This contradicts that v is an end-vertex. Therefore, $v \in S$. Thus v is contained in every detour set of G.

(ii) Let S be the square free detour basis of G. If S' consists of all the end-vertices of G, then by (i) $dn_{\Box f}(G) = |S| \geq |S'|$. If S' is a square free detour set of G, then $dn_{\Box f}(G) \leq |S'|$. Hence $dn_{\Box f}(G) = |S'|$. Then $S = S'$ is the square free detour basis of G. Thus the uniqueness of the square free detour basis containing end-vertices is proved.

Corollary 2.7 For any tree T with l end-vertices, $dn_{\Box f}(T) = l$.

Corollary 2.8 If G is a connected graph with l end-vertices, then $\{2, l\} \leq dn_{\Box f}(G) \leq n$.

Theorem 2.9 If S is a square free detour set of G, then for any cut-vertex y of G, every component of $G - y$ consists of a vertex of S.

Proof. Let S be a square free detour set and F be a component of $G - y$. Suppose that F does not contain any element of S. Let $x \in F$. Let $u, v \in S$ such that x lies on any $u - v$ square free detour path P^* in G for two vertices u and v different from x. Then the $u - x$ subpath Q' of $P^*$and $x - v$ subpath R' of P^* contain the cut-vertex y of G, which implies P^* is not a path. This contradicts our assumption. Hence the proof.

Corollary 2.10 Let y be a cut-vertex in G and let the number of components of $G - y$ be t. Then $dn_{\Box f}(G) \geq t$.

Corollary 2.11 Let y be a cut-vertex in G. Then a vertex of the square free detour set S belongs to every branch at y.

Theorem 2.12 If S is a square free detour basis of G, then no cut-vertex of G belongs to S.

Proof. Consider a square free detour basis S of G and a cut-vertex y such that $y \in S$. Then every component of $G - y$ contains a vertex of S, by Theorem 2.9. Suppose F and H are two components of $G - y$. Then y is an internal vertex of all $u - v$ square free detour paths, where $u \in F$ and $v \in H$. Let $S^* = S - \{y\}$. Clearly, S^* is a square free detour set of G. Hence $|S^*| < |S|$ which contradicts that S is a square free detour basis of G.

Theorem 2.13 For a non-complete connected graph G of order n with vertex connectivity κ, $dn_{\Box f}(G) \leq n - \kappa$.

Proof. Let G be a non-complete connected graph. Then $1 \leq \kappa \leq n - 2$. Let $A' = \{a_1, a_2, a_3, \ldots, a_\kappa\}$ be a vertex cut of G. Suppose F_1, F_2, $F_3, \ldots, F_t (t \geq 2)$ are the components of $G - A'$ and $S = V - A'$. Then every vertex of A' is adjacent to at least one vertex of $G_i (1 \leq i \leq t)$. By Theorem 2.12, $a_j \notin S$ $(1 \leq j \leq \kappa)$ and so $dn_{\Box f}(G) \leq n - \kappa$.

Remark 2.14 For the cycle graph C_4, $\kappa = 2$. Hence $dn_{\Box f}(G) = n - \kappa$. Thus the bounds given in Theorem 2.13

hold sharp. Also, For the graph G_2 depicted in Figure 2 with $n = 8,\ \kappa = 1,\ dn_{\square f}(G) < n - \kappa$. Thus the bounds given in Theorem 2.13 hold strict.

We denote the union of the m disjoint copies of G by $mG(m \geq 1)$.

Theorem 2.15 If $G = \left(K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_t} \cup mK_1 \right) + x$ is a block graph of order $n \geq 4$ with $m \geq 1$ and $n_1 + n_2 + \cdots + n_r + m = n - 1$, then $dn_{\square f}(G) = t + m$.

Proof. Consider $G = \left(K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_t} \cup mK_1 \right) + y$, a block graph. Let $x_1, x_2, x_3, \ldots, x_m$ be the end-vertices of G and S be a square free detour set of G. Then by Theorem 2.6, $x_j \in S(1 \leq j \leq m)$ and by Theorem 2.9, S consists of an element from each $K_{n_j}(1 \leq j \leq t)$. Let exactly one element y_j from each component K_{n_j} be chosen such that $y_j \in S(1 \leq j \leq t)$. Since every square free detour joining a pair of vertices of S contains the element y of G and by Theorem 2.14, $y \notin$ S. Hence S is the square free detour set with $t + m$ vertices and $dn_{\square f}(G) = t + m$.

Theorem 2.16 Let $G = (V, E)$ be a complete graph $K_n(n \geq 2)$. Then a set of vertices S is a square free detour basis of G if and only if S contains any two adjacent vertices of G.

Proof. Let $G = K_n$ be a complete graph of order $n(n \geq 2)$ and $V(G) = \{ x_1, x_2, x_3, \ldots, x_n \}$. Let $S = \{x_1, x_2\}$ be a set of two vertices of G. Let $x_i \in V$.

Case 1: Let $x_i \notin$ S . Then $x_i\ (3 \leq i \leq n)$ lies on a square free detour $x_1 x_i x_2$ of length 2.

Case 2: Let $x_i \in$ S and say $x_i = x_1$. Then x_i lies on a square free detour $x_i\, x_j\, x_2$ of length 2 where $x_j \notin$ S. Thus every vertex x_i of V lies on a square free detour in G and so S is a square free detour set of G. Also, $|S| = 2$. Therefore, S is a square free detour basis of G.

Conversely, let S be a square free detour basis of G. Let S^*be any set containing two vertices that are adjacent in G. Then by previous discussion of this theorem, S^* is a square free detour basis of G. Hence $|S| = |S^*| = 2$. Thus S contains any two adjacent vertices of G.

Theorem 2.17 Let $G = (V,\ E)$ be a complete bipartite graph $K_{n_1,n_2}(2 \leq n_1 \leq n_2)$ with partitions X and Y where $|X| = n_1, |Y| = n_2$. Then a set $S \subseteq V$ is a square free detour basis of G if and only if $S = X$.

Proof. Let $G = K_{n_1,n_2}(2 \leq n_1 \leq n_2)$ be a complete bipartite graph with bipartite sets X and Y. Let $X = \left\{ x_1, x_2, x_3, \ldots, x_{n_1} \right\}$ and $Y = \{ y_1, y_2, y_3, \ldots, y_{n_2} \}$. Let $S = X$ and $v \in V$.

Case 1: Let $v \in X$. Let $v = x_k(1 \leq k \leq n_1)$. Then $v = x_i$ lies on the square free detour $v = x_k x_l x_m$ for some distinct k and m $(1 \leq k, m \leq n_1,\ 1 \leq l \leq n_2)$, where $D_{\square f}(x_k,\ x_m) = 2$.

Case 2: Let $v \in Y$. Let $v = y_l(1 \leq j \leq n)$. Then $v = y_l$ lies on the square free detour $v = x_k\, y_l\, x_m (1 \leq k, m \leq n_1,\ 1 \leq l \leq n_2,\ k \neq m)$ such that $D_{\square f}(x_k,\ x_m) = 2$.

Hence every vertex of V lies on the square free detour. Thus S is a square free detour set of G. Moreover, $|S| = n_1$ and so S is a square free detour basis of G.

Conversely, let S be a square free detour basis of G. Let $S' \subseteq V$ and S' contain the elements from both X and Y. Let $x \in X$ and $y \in Y$. Then any $x - y$ detour induces a square. Therefore, S' cannot contain the elements of both X and Y. Hence S' must consist of the vertices of X or that of Y. Since $|Y| \geq |X|$, S' consists of n_1 vertices of X only. Then by the previous discussion of this theorem, S' is a square free detour basis of G. Therefore, $|S| = |\, S'| = n_1$ and $S = X$.

Theorem 2.18 Let $G = (V,\ E)$ be a cycle C_n of order $n(n \geq 3; n = \text{ odd })$. Then a set $S \subseteq V$ is a square free detour basis of G if and only if S contains any two vertices adjacent to each other in G.

Proof. Let $G = C_n: x_1, x_2, x_3, \ldots, x_n, x_1$ be a cycle of order $n(n \geq 3; n = \text{ odd })$. Let $S = \{x_i, x_{i+1}: 1 \leq i \leq n - 1\}$ be a set of two adjacent vertices of G. Then all the vertices of G lie on the square free detour $x_i - x_{i+1}$ of length $n - 1$ and so S is a square free detour set of G. Moreover, $|S| = 2$. Hence S is a square free detour basis of G.

Now assume that S is a square free detour basis of G. Suppose $S^*$is a set consists of two vertices adjacent in G. Then by previous discussion of this theorem, S^* is a square free detour basis of G. Therefore, $|S| = |\, S^*| = 2$. Thus S contains any two vertices adjacent in G.

Theorem 2.19 Let $G = (V, E)$ be a cycle $n(n \geq 6; n = \text{even})$. Then a set $S \subseteq V$ is a square free detour basis of G if and only if S consists of two vertices either adjacent or antipodal to each other in G.

Proof. Suppose $G = C_n: x_1, x_2, x_3, \ldots, x_n, x_1$ is an even cycle of order $n \geq 6$.

Case 1: Consider $S = \left\{x_j, x_k \left| 1 \leq j \leq n;\ x_k \in N(x_j) \right.\right\}$, a set of two vertices that are adjacent in G. Then all the vertices of G lie on the square free detour $x_j - x_k$ of length $n - 1$.

Case 2: Suppose $S = \{x_i, x_{i+\frac{n}{2}} \mid 1 \leq i \leq \frac{n}{2}\}$ is a set of two antipodal vertices of G. Then there exist two square free detours $x_i - x_{i+\frac{n}{2}}$ and $x_{i+\frac{n}{2}} - x_i$ of length $\frac{n}{2}$. Obviously, each $x_j(1 \leq j \leq n)$ of $V(G)$ lies on any one of these square free detours. Thus S is a square free detour set of G. Since $|S| = 2, S$ is a square free detour base of G.

Now assume in a graph G, S is a square free detour basis. Let S^* be any set of two vertices that are either adjacent or antipodal in G. By previous discussion of this theorem, $S^*$is a square free detour basis of G. Hence $|S| = |S^*| = 2$. Thus S contains two vertices either adjacent or antipodal in G.

Theorem 2.20 Let G be a cycle C_4. Then a set $S \subseteq V$ is a square free detour basis of G if and only if S contains any two vertices antipodal to each other in G.

Proof. Let $G = K_4$ or C_4, where $V(G) = \{x_1, x_2, x_3, x_4\}$. Suppose $S = \{x_s, x_t\}$ is the set of two vertices antipodal in G with $D_{\square f}(x_s, x_t) = 2$. Then each vertex of G lies on $x_s{-}x_t$ square free detour. Thus S is a square free detour set of G. Moreover $|S| = 2$. Therefore, S is a square free detour basis of G.
Assume that S is a square free detour basis of G. Let S^* be a set of any two vertices that are antipodal in G. By previous discussion of this theorem, S^* is a square free detour basis of G. Therefore, $|S| = |S^*| = 2$ and S contains any two antipodal vertices to each other in G.

Theorem 2.21 Let $G = (V, E)$ be a wheel $W_n = K_1 + C_{n-1}(n \geq 11$ and n is odd). Then a set $S \subseteq V$ is a square free detour basis of G if and only if S contains any two vertices either adjacent or antipodal in C_{n-1} and the hub.
Proof. Let G be a wheel $W_n = K_1 + C_{n-1}(n \geq 11$ and n is odd). Let $V(K_1) = \{x_0\}$. Let $S^* = \{y, z\}$ be the set of two vertices of C_{n-1} such that y and z are either adjacent or antipodal on C_{n-1}. Then by Theorem 2.19, all the vertices of G except the hub lie on a $y - z$ square free detour. Thus $S = S^* \cup \{x_0\}$ is a square free detour set of G. Thus $|S| = 3$ and so S is a square free detour basis of G.

Now consider S is a square free detour basis of G. Assume that $S^{\#}$ be any set contain the hub with two vertices either adjacent or antipodal on C_{n-1}. By previous discussion of this theorem, $S^{\#}$ is a square free detour basis of G. Thus $|S| = |S^{\#}| = 3$ and so S contains any two vertices that are either adjacent or antipodal in C_{n-1} and the hub.

Theorem 2.22 Let G be a wheel $W_n = K_1 + C_{n-1}(n = 6, n \geq 10$ and n is even). Then a set $S \subseteq V$ is a square free detour basis of G if and only if S contains any two vertices that are adjacent in C_{n-1} and the hub.
Proof. Let G be a wheel $W_n = K_1 + C_{n-1}(n \geq 10$ and n is even). Let $V(K_1) = \{x_0\}$. Let $S_1 = \{u, v\}$ be the set of two vertices of C_{n-1} such that u and v are either adjacent or antipodal on C_{n-1}. Then by Theorem 2.18, all the vertices except the hub of G lie on the $u - v$ square free detour. Thus $S = S_1 \cup \{x_0\}$ is a square free detour set of G. Moreover, $|S| = 3$. Therefore, S is a square free detour basis of G.

Conversely, let S be a square free detour basis of G. Suppose S^* be a set containing any two vertices that are adjacent on C_{n-1} and a hub. By previous discussion of this theorem, S^* is a square free detour basis of G. Therefore, $|S| = |S^*| = 3$. Thus S contains any two vertices that are adjacent in C_{n-1} and the hub of W_n.

Theorem 2.23 Let $G = (V, E)$ be a wheel $W_n = K_1 + C_{n-1}(n = 4,5,7,8,9)$. Then a set of vertices S is a square free detour set of G if and only if S contains any two vertices u and v such that

(i) u and v are adjacent when $n = 4$

(ii) u and v are antipodal when $n = 5$

(iii) $D_{\square f}(u, v) = 4$ when $n = 7,8,9$.

Proof. (i) Let $G = W_4$ be the wheel with central vertex x_0. Since $W_4 \cong K_4$, the result follows from Theorem 2.20.
(ii) Let $G = W_5$, where x_0 is the central vertex. Suppose $S = \{u, v\}$ is the set of two vertices antipodal on C_4 with $D_{\square f}(u, v) = 2$. Then by Theorem 2.20, all the vertices of C_4 lie on a $u - v$ square free detour. Moreover, the central vertex of W_5 also lies on u, x_0, v square free detour. Thus S is a square free detour set of G. Since $|S| = 2, S$ is a square free detour basis of W_5.

Assume that S is a square free detour basis of W_5. Let S^* be a set of any two vertices that are antipodal on C_4. Then from the previous discussion S^* is a square free detour basis of W_5. Therefore, $|S| = |S^*| = 2$ and so S contains any two antipodal vertices of C_4.
(iii) Let $G = W_n$ $(6 \leq n \leq 9)$, where x_0 is the central vertex. Suppose $S = \{u, v\}$ is the set of two vertices on C_{n-1} such that $D_{\square f}(u, v) = 4$. Then all the vertices of G lie on $u - v$ square free detour. Thus S is a square free detour set of W_n. Since $|S| = 2, S$ is a square free detour basis of $W_n(n = 7,8,9)$.

Assume that S is a square free detour basis of $W_n(n = 7,8,9)$. Consider S^* is a set of any two vertices that are at square free detour distance 4 on C_{n-1}. By previous discussion of this theorem, S^* is a square free detour basis of $W_n(n = 7,8,9)$. Therefore, $|S| = |S^*| = 2$ and S contains any two vertices on C_{n-1} with $D_{\square f} = 4$.

Theorem 2.24 Let $G = (V, E)$ be a Windmill $W_n^{(m)}$consisting of m copies of $K_n(m \geq 2)$ with a vertex x in common. Then the set of vertices S is a square free detour basis of G if and only if S consists of m vertices, exactly one vertex adjacent to x from each copy of K_n.
Proof. Suppose $G = W_n^{(m)}$ is a Windmill containing m copies of $K_n(n \geq 2)$ with the common vertex x and of order $m(n-1)+1$. Let $S = \{x_{kl} | 1 \leq k \leq m;\ 1 \leq l \leq n-1\}$ be a set of m vertices adjacent to x, exactly one from m copies of $K_n^{(m)}$. Then every vertex of G lies on any square free detour $x_{il} - x_{(i+1)l}(1 \leq i \leq m-1;\ 1 \leq l \leq n-1)$ of length 2. Thus $|S| = m$ and S is a square free detour basis of $W_n^{(m)}$.

Now let S be a square free detour basis of $W_n^{(m)}$. Suppose S^* is a set of m vertices of G, taken exactly one vertex from m copies of K_n. By previous discussion of this theorem, S^* is a square free detour basis of $W_n^{(m)}$. Thus $|S| = |S^*| = m$ and S contains exactly one vertex from each $K_n^{(i)}(1 \leq i \leq m)$ of $W_n^{(m)}$.

Theorem 2.25 Let $G = (V, E)$ be a Dutch Windmill $D_n^{(m)}(n \geq 3, m \geq 2)$ consisting of m copies C_n with a common vertex x. Then a set of vertices S is a square free detour basis of G if and only if S contains m vertices exactly one from each copy of $C_n(n \geq 3)$ in $D_n^{(m)}$.
Proof. Let $G = D_n^{(m)}$ be a Dutch Windmill graph of order $m(n-1)+1$ consisting of m copies of $C_n(n \geq 3)$ with the

common vertex x. Let $V(D_n^{(m)}) = \{x, x_{kl} | 1 \leq k \leq m;\ 1 \leq l \leq n-1\}$. Let a set S consist of m vertices of $D_n^{(m)}$. Then we have three cases.

Case 1: Consider $n \geq 3$ and n is odd. Suppose $S = \{x_{kj} | 1 \leq k \leq m;\ l = 1\ or\ (n-1)\}$ is a set of m adjacent vertices of x, exactly one vertex from each copy of C_n. Then all the vertices of G lie on the square free detour joining two vertices of S, which admits x as the central vertex with $D_{\square f} = 2(n-1)$ and $D_{\square f}(x, x_{kj}) = n-1$. Since x is a cut-vertex of G by Theorem 2.9, every component of $G - x$ contains a vertex of S and so S is a square free detour set of G.

Case 2: Consider $n \geq 6$ and n is even. Suppose $S = \{x_{ij} | 1 \leq i \leq m; j = 1\ or\ \frac{n}{2}\ or\ (n-1)\}$ contains either m vertices that are adjacent or antipodal to x. Then every vertex of G lies on $x_{kj} - x_{(k+1)j}$ $(1 \leq k \leq m-1; j = 1\ or\ \frac{n}{2}\ or\ (n-1))$ square free detour. Consider $u = x_{kj}$ and $v = x_{(k+1)j}$. We have three subcases.

Subcase 1: Let S contain the vertices adjacent to x and let $j = 1\ or (n-1)$. Then $D_{\square f}(u, x) = D_{\square f}(v, x) = j$ and all the vertices of $D_n^{(m)}$ lie on a $u - v$ square free detour of length $2(n-1)$. Thus S is a square free detour set of $D_n^{(m)}$.

Subcase 2: Let S contain the vertices antipodal to x and let $j = \frac{n}{2}$. Then $D_{\square f}(u, x) = D_{\square f}(v, x) = j$ and all the vertices of $D_n^{(m)}$ lie on a $u - v$ square free detour of length n. Therefore, S is a square free detour set of $D_n^{(m)}$.

Subcase 3: Let S contain the vertices either adjacent or antipodal to x. Without loss of generality consider $u = x_{k(n-1)}$ and $v = x_{(k+1)\frac{n}{2}}$. Then $D_{\square f}(u, x) = n - 1$, $D_{\square f}(v, x) = \frac{n}{2}$ and all vertices of $D_n^{(m)}$ lie on a $u - v$ square free detour of length $\frac{3n-2}{2}$. Thus S is a square free detour set of $D_n^{(m)}$.

Case 3: Let $n = 4$ and S consist of m antipodal vertices of x, from each copy of $C_4^{(i)}: x_{i1}, x_{i2}, x_{i3}, x_{i4}, x_{i1} (1 \leq i \leq m)$. Then every vertex of G lies on $x_{k2} - x_{(k+1)2} (1 \leq k \leq m - 1)$ square free detour of length 4 where $D_{\square f}(x, x_{k2}) = D_{\square f}(x, x_{(k+1)2}) = 2$. Thus S is a square free detour set of $D_n^{(m)}$.

From all the above three cases, we observe that $|S| = m$ and S is a square free detour basis of G.

Assume that S is a square free detour basis of $D_n^{(m)}$. Consider S^* is any set of m vertices adjacent to the common vertex x, exactly one vertex from each copy of C_n of G when n is odd $(n \geq 3)$, m antipodal vertices of x when n is 4 and either m adjacent vertices or antipodal vertices of x when n is even $(n \geq 6)$. Then by previous discussion of this theorem, S^* is a square free detour basis of $D_n^{(m)}$. Thus $|S| = |S^*| = m$ and S contains m vertices that are adjacent or antipodal to x or both adjacent or antipodal to x according to n is odd, n is 4 and n is even, exactly one element from each copy of $C_n (n \geq 3)$ in $D_n^{(m)}$.

Corollary 2.26

(a) For a tree T with l end-vertices, $dn_{\square f}(T) = l$

(b) For a complete graph K_n, $dn_{\square f}(K_n) = 2$

(c) For a complete bipartite graph $K_{n_1,n_2} (2 \leq n_1 \leq n_2)$, $dn_{\square f}(K_{n_1,n_2}) = n_1$

(d) For a cycle $C_n (n \geq 3)$, $dn_{\square f}(C_n) = 2$.

(e) For a wheel $W_n = K_1 + C_{n-1}$, $dn_{\square f}(W_n) = \begin{cases} 2 & \text{if } n = 4,5,7,8,9 \\ 3 & \text{if n} = 6, n \geq 10 \end{cases}$

(f) For a Windmill $W_n^{(m)}$, $dn_{\square f}(W_n^{(m)}) = m$

(g) For a Dutch Windmill $D_n^{(m)}$, $dn_{\square f}(D_n^{(m)}) = m$.

Proof. (a) This follows from Corollary 2.7

(b) This follows from Theorems 2.16 and 2.20

(c) This follows from Theorem 2.17

(d) This follows from Theorems 2.18, 2.19 and 2.20

(e) This follows from Theorems 2.21, 2.22 and 2.23

(f) This follows from Theorems 2.24

(g) This follows from Theorem 2.25.

Theorem 2.27 For each pair of integers p and n with $2 \leq p \leq n$, there exists a connected graph G of order n with $dn_{\square f}(G) = p$.

Proof. Suppose that G is a connected graph of order n.

Case 1: $p = n = 2$. It is trivially true for complete graph K_2 and path P_2.

Case 2: $2 \leq p < n$. Assume that P is a path of order $n - p + 2$. Then the graph G obtained from P by adding $p - 2$ new vertices to P and joining them to any cut-vertex of P is a tree of order n and so by corollary 2.7, $dn_{\square f}(G) = p$.

Theorem 2. 28 For a connected graph $G = (V, E)$ of order n, $2 \leq dn(G) \leq dn_{\square f}(G) \leq dm(G) \leq n$.

Proof.

Case 1: Suppose $G = T$ is a tree. Then T is acyclic and every square free detour set is a detour set and a detour monophonic set. Hence $dn(G) = dn_{\square f}(G) = dm(G)$.

Case 2: Assume $G = C$ is a cyclic graph and C^* is a cycle in G. Suppose xy is a chord of G. Let a and b be two vertices different from x and y in C^* such that these four vertices induce a square. Then a vertex a or b must lie in a square free detour set. Hence $dn(C) \leq dn_{\square f}(C) = dm(G)$. If no four vertices of C^* induce a square, then $dn_{\square f}(G) \leq dm(G)$. Since every square free detour set is a detour set $dn(C) \leq dn_{\square f}(C)$. Also, since C is connected $dm(C) \leq n$. Hence $2 \leq dn(C) \leq dn_{\square f}(C) \leq dm(C) \leq n$.

Remark 2.29 The bounds given in Theorem 2. 29 hold sharp for the path P_2, $dn(P_2) = dn_{\square f}(P_2) = dm(P_2) = 2.$ Moreover, the inequalities given in Theorem 2. 29 hold strict for the graph G_2 given in Figure 2 with order $n =$

$8, dn(G_2) = 2, dn_{\square f}(G_2) = 3$ and $dm(G_2) = 4$. Hence $dn(G_2) < dn_{\square f}(G_2) < dm(G_2) < n$.

Theorem 2.30 For any three integers α, β and γ with $3 \leq \alpha \leq \beta \leq \gamma$, there exists a connected graph G such that $dn(G) = \alpha, dn_{\square f}(G) = \beta$ and $dm(G) = \gamma$.

Proof. Let G_1 be a graph obtained from the path $P_5: x_1, x_2, x_3, x_4, x_5$ of order 5 by the addition of $\alpha - 1$ new vertices $y_1, y_2, \dots, y_{\alpha-1}$ and by joining each y_a $(1 \leq a \leq \alpha - 1)$ to the vertex x_5 in P_5. Suppose G_2 is the graph generated from G_1 by addition of $2(\beta - \alpha)$ new vertices $p_1, p_2, \dots, p_{\beta-\alpha}$ and $q_1, q_2, \dots, q_{\beta-\alpha}$ and by joining each $p_b (1 \leq b \leq \beta - \alpha)$ to the vertex x_2 in P_5 and joining each vertex $q_b (1 \leq b \leq \beta - \alpha)$ to the vertex x_3 in P_5. Let $P_3^c: r_c, s_c, t_c$ $(1 \leq c \leq \gamma - \beta)$ be the $\gamma - \beta$ copies of P_3. Let G_3 be obtained from the graph G_2 by adding new vertices $r_1, r_2, \dots, r_{\gamma-\beta}$, $s_1, s_2, \dots, s_{\gamma-\beta}$ and $t_1, t_2, \dots, t_{\gamma-\beta}$ and joining each vertex $r_c (1 \leq c \leq \gamma - \beta)$ to the vertex x_3 in P_5 and joining each vertex t_c $(1 \leq c \leq \gamma - \beta)$ to the vertex x_4 in P_5. The required graph $G = G_3$ is a connected graph of order $3\gamma - \beta - \alpha + 4$ and is shown in Figure 3.

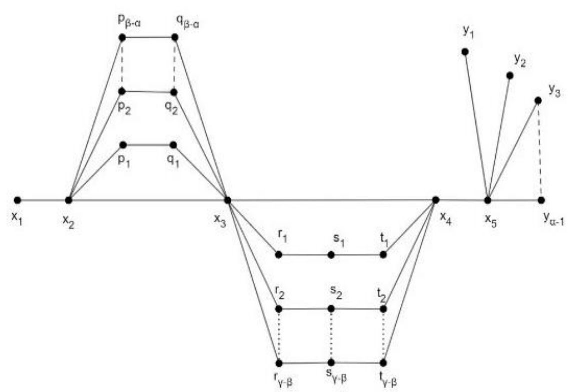

FIGURE 3: G

By Theorem 2.6, it can be easily verified that $S_1 = \{x_1, y_1, y_2, y_3, \dots, y_{\alpha-1}\}$ is a detour basis of G, $S_2 = S_1 \cup \{p_1, p_2, \dots, p_{\beta-\alpha}\}$ is a square free detour basis and $S_3 = S_2 \cup \{t_1, t_2, \dots, t_{\gamma-\beta}\}$ is a detour monophonic basis. Hence $dn(G) = \alpha, dn_{\square f}(G) = \beta$ and $dm(G) = \gamma$ and $\alpha \leq \beta \leq \gamma$.

3. CONCLUSION

In this article, we determined the square free detour number of some standard graphs and special graphs. The relationship between the square free detour number and detour number was discussed. Derivation of similar results in this context for some other classes of graphs is an open area of research.

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