Mathematical solution of generalized reaction diffusion model in blow up situation

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Abstract :

This paper is associated with the mathematical solution of generalized reaction diffusion model in blow up situation. To show its capability and efficiency, some examples are also illustrated. To find the mathematical solution of thus generated generalized reaction diffusion problem, we use some integral transform techniques as well as some methods related with ordinary and partial differential equation. The effectiveness of this method is also increased when we use the blow up condition.

Keywords: Reaction diffusion equation, blow up situation, integral transform, Holder's inequality, differential equation, partial differential equation

1. Introduction

In some past years, the interest of researcher from various fields is continuously increasing in application and solution of reaction diffusion equation. The some applications are in which it take place are mathematical biology, environment and ecology, life sciences, chemical reaction. Whenever reaction diffusion is of generalized nature then it is more useful then ordinary reaction diffusion equation. In this paper we are taking generalized reaction diffusion equation which have dual nature linear and non-linear reaction diffusion equation. We are also adding one more condition to it and it is blow up condition. The review literature regarding previous research in this area is here listed in brief. Kapoor [1] explained the mathematical model's in biology and medicine. Kumar et al. [2] focused on exact and numerical solution of non-linear reaction diffusion equation by using Cole-Hoff transformation Alvero et al. [3] briefed reaction – diffusion equations. A chemical application. Kumar et al. [4] designed a mathematical model to solve reaction diffusion equation by using Homotopy Perturbation method. Kuttler [5] illustrated reaction diffusion with applications. Bhadauria et al. [6-7] formulated a mathematical model to solve reaction diffusion equation using differential transform method and solution of reaction diffusion equation by Adomain Decomposition method. Polyanin [8] explained non-linear reaction-diffusion equations with delay. Some the organs, test problems, exact and numerical solutions. Kaltenbacher et al. [9] focused on the identification of a non-linear term in a reaction-diffusion equation. Singh et al. [10] worked on mathematical analysis of reaction diffusion equation with ecological parameters.

2. Solution of the Problem

Partial differential equation for generalized reaction diffusion model in blow up situation is –

$$\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} + Ru\left(u^{p-1} - 1\right), \text{ where } p > 1 \qquad (1)$$

under the boundary conditions

$$\left.\begin{aligned} &(i)\ u(0,t) = 0 \\ &(ii)\ u(\pi,t) = 0 \\ &(iii)\ u(x,0) = f(x) \end{aligned}\right\} \qquad (2)$$

Multiplying equation (1) by sin nx both sides

$$\frac{\partial u}{\partial t}\sin nx = D\frac{\partial^2 u}{\partial x^2}\sin nx + R\left[u^p \sin nx - u \sin nx\right] \tag{3}$$

Integrating above equation with respect to x under the limit 0 to π

$$\int_0^\pi \frac{\partial u}{\partial t}\sin nx\, dx = D\int_0^\pi \frac{\partial^2 u}{\partial x^2}\sin nx\, dx + R\left[\int_0^\pi u^p \sin nx\, dx - \int_0^\pi u \sin nx\, dx\right] \tag{4}$$

let $g = g(t) = \int_0^\pi u \sin nx\, dx$ (5)

Differentiating above equation with respect to t, by Leibnitz rule (differentiation under the sign of integration)

$$\frac{dg}{dt} = \int_0^\pi \frac{\partial u}{\partial t}\sin nx\, dx \tag{6}$$

By equation (4), (5) and (6)

$$\frac{dg}{dt} = D\left[\left(\sin nx \frac{\partial u}{\partial x}\right)_0^\pi - n\int_0^\pi \cos nx \frac{\partial u}{\partial x} dx\right] + R\left[\int_0^\pi u^p \sin nx\, dx - g\right] \tag{7}$$

$$\frac{dg}{dt} = D\left[0 - 0 - n\left(\cos x\, u(x,t)\right)_0^\pi - n^2\int_0^\pi u \sin nx\, dx\right] + R\left[\int_0^\pi u^p \sin nx - g\right] \tag{8}$$

$$\frac{dg}{dt} = -\left(R + Dn^2\right)g + R\int_0^\pi u^p \sin nx\, dx \tag{9}$$

$$\int_0^\pi u^p \sin nx\, dx = \int_0^\pi (u \sin nx)^p (\sin nx)^{1-p} dx$$

By Holder inequality

$$\left|\int_0^\pi u^p \sin nx\, dx\right| \le \left(\int_0^\pi u \sin nx\right)^p \left(\int_0^\pi \sin nx\, dx\right)^{1-p}$$

$$= g^p\left[\left(-\frac{\cos nx}{n}\right)^\pi\right]^{1-p}$$

$$= g^p\left(\frac{1-\cos n\pi}{n}\right)^{1-p}$$

In blow of condition

$$\int_0^\pi u^p \sin nx\, dx = \begin{cases} g^p\left(\frac{2}{n}\right)^{1-p} & \text{if } n \text{ is odd integer} \\ 0 \text{ if } n \text{ is even integer} \end{cases} \tag{10}$$

Case – I: If n is odd integer by 9 and (10) (i)

$$\frac{dg}{dt} = -\left(R + Dn^2\right)g + \frac{2R}{n}\left(\frac{gn}{2}\right)^p \tag{11}$$

$$\frac{dg}{dt} + \left(R + Dn^2\right)g = \frac{2R}{n}\left(\frac{gn}{2}\right)^p \tag{12}$$

Dividing by g^p both sides

$$\frac{1}{g^p}\frac{dg}{dt} + \left(R + Dn^2\right)\frac{1}{g^{p-1}} = \frac{2R}{n}\left(\frac{n}{2}\right)^p \tag{13}$$

Let $\frac{1}{g^{p-1}} = f, \frac{(1-p)}{g^p}\frac{dg}{dt} = \frac{df}{dt}$ (14)

By (13) and (14)

$$\frac{1}{(1-p)}\frac{df}{dt} + \left(R + Dn^2\right)f = \frac{2R}{n}\left(\frac{n}{2}\right)^p \tag{15}$$

Multiplying by (1 – p) both sides

$$\frac{df}{dt} + \left(R + Dn^2\right)(1-p)f = \frac{2R(1-p)}{n}\left(\frac{n}{2}\right)^p \tag{16}$$

Above is a linear differential equation in f and t

Integrating factor is $e^{\int \left(R+Dn^2\right)(1-p)dt}$

I.F. $= e^{\left(R+Dn^2\right)(1-p)t}$ (17)

Solution of (16) is given by

$$e^{\left(R+Dn^2\right)(1-p)t} f = \int \frac{2R(1-p)}{n}\left(\frac{n}{2}\right)^p e^{\left(R+Dn^2\right)(1-p)t} dt + c_1 \tag{18}$$

$$e^{\left(R+Dn^2\right)(1-p)t} f = \frac{\frac{2R(1-p)}{n}\left(\frac{2}{n}\right)^p e^{\left(R+Dn^2\right)(1-p)t}}{\left(R + Dn^2\right)(1-p)} + c_1 \tag{19}$$

$$f = \frac{2R}{n\left(R + Dn^2\right)}\left(\frac{n}{2}\right)^p + c_1 e^{\left(R+Dn^2\right)(p-1)t} \tag{20}$$

$$\frac{1}{g^{p-1}} = \frac{2R}{n\left(R + Dn^2\right)}\left(\frac{n}{2}\right)^p + c_1 e^{\left(R+Dn^2\right)(p-1)t}$$

$$g = \left[\frac{1}{\frac{2R}{n\left(R + Dn^2\right)}\left(\frac{n}{2}\right)^p + c_1 e^{\left(R+Dn^2\right)(p-1)t}}\right]^{\frac{1}{p-1}} \tag{21}$$

$$\int_0^{\pi} u \sin nx \, dx = \left[\frac{1}{\frac{2R}{n(R+Dn^2)}\left(\frac{n}{2}\right)^p + c_1 e^{(R+Dn^2)(p-1)t}} \right]^{\frac{1}{p-1}} \quad (22)$$

Now

$$u(x,t) = \sum_{n=1 1,3,5\ldots} \left[\frac{1}{\frac{2R}{n(R+Dn^2)}\left(\frac{n}{2}\right)^p + c_1^{(R+Dn^2)(p-1)t}} \right]^{\frac{1}{p-1}} \sin nx \quad (23)$$

Taking t = 0

$$f(x) = u(0,t) = \sum_{n=1,\,3,\,5\ldots} \left(\frac{1}{\frac{2R}{n(R+Dn^2)}\left(\frac{n}{2}\right)^p + c_1} \right)^{\frac{1}{p-1}} \sin nx \quad (24)$$

$$\left(\frac{1}{\frac{2R}{n(R+Dn^2)}\left(\frac{n}{2}\right)^p + c_1} \right)^{\frac{1}{p-1}} = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx \quad (25)$$

Case – II when n is even integer

$$\frac{dg}{dt} = -(R+Dn^2)g$$

By separation of variable

$$\frac{dg}{g} = -(R+Dn^2)dt$$

Integrating

$$\log g = -(R+Dn^2)t + \log c_2$$

$$g = c_2 \bar{e}^{(R+Dn^2)t} \quad (26)$$

$$\int_0^{\pi} u(x,t) \sin nx \, dx = c_2 \bar{e}^{(R+Dn^2)t} \quad (27)$$

$$u(x,t) = \sum_{n=2,4,6} c_2 \bar{e}^{(R+Dn^2)t} \sin nx \quad (28)$$

Taking t = 0

$$f(x) = u(x,0) = \sum_{n=2,4,6} c_2 \sin\ nx$$

$$c_2 = \frac{2}{\pi}\int_0^{\pi} f(x) \sin\ dx \tag{29}$$

Combining both of the aspects of n, we can say that solution is given by

$$u(x,t) = \sum_{n=1,3,4} \left[\frac{1}{\frac{2R}{n(R+Dn^2)}\left(\frac{n}{2}\right)^p + c_1^{(R+Dn^2)(p-1)t}} \right]^{\frac{1}{p-1}} \sin nx$$

$$+ \sum_{n=2,4,6} c_2 \bar{e}^{(R+Dn^2)(p-1)t} \sin\ nx \tag{30}$$

where

c_1 and c_2 are obtained by

$$\left[\frac{1}{\frac{2R}{n(R+Dn^2)}\left(\frac{n}{2}\right)^2 + c_1} \right]^{\frac{1}{p-1}} = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\ dx \tag{31}$$

and $c_2 = \frac{2}{\pi}\int_0^{\pi} f(x) \sin\ nx\ dx$ (32)

Example: 1

Let the initial value of function u (x,t) in the solution of generalized reaction diffusion problem is given by u (x, 0) = f (x) = 1

The solution is given by equation (30) and c_1 is obtained by

$$\left(\frac{1}{\frac{2R}{n(R+Dn^2)}\left(\frac{n}{2}\right)^p + c_1} \right)^{\frac{1}{p-1}} = \frac{2}{\pi}\int_0^{\pi} \sin nx\ dx \text{ for n being odd integer}$$

$$= \frac{2}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi} = \frac{4}{\pi n}$$

$$\left(\frac{1}{\frac{2R}{n(R+Dn^2)}\left(\frac{n}{2}\right)^p + c_1} \right)^{\frac{1}{p-1}} \frac{4}{\pi n} \tag{33}$$

c_2 is obtained by

$$c_2 = \frac{2}{\pi}\int_0^{\pi} \sin dx = \frac{2}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi} = 0 \text{ for being even integer}$$

$$c_2 = 0 \tag{34}$$

Hence the solution of generalized reaction diffusion problem with above specific boundary condition is obtained by replacing c_1 and c_2 by (33) and (34) in equation (30)

Example – 2

Let the initial value of function u (x, t) in the solution of generalized reaction diffusion problem is given by u (x, 0) = f (x) = e^x

The solution is given by equation (30) and

c_1 is obtained by

$$\left(\frac{1}{\frac{2R}{n(R+Dn^2)}\left(\frac{n}{2}\right)^p + c_1}\right)^{\frac{1}{p-1}} = \frac{2}{\pi}\int_0^{\pi} e^x \sin nx\, dx \text{ for n being odd integer}$$

$$= \frac{2}{\pi}\left(\frac{e^x}{1+n^2}(\sin nx - n\cos nx)\right)_0^{\pi}$$

$$= \frac{2n}{\pi(1+n^2)}\left[e^{\pi} - 1\right]$$

Hence $$\left(\frac{1}{\frac{2R}{n(R+Dn^2)}\left(\frac{n}{2}\right)^p + c_1}\right)^{\frac{1}{p-1}} \frac{2n}{\pi(1+n^2)}\left(e^{\pi} - 1\right) \tag{35}$$

c_2 is obtained by

$$c_2 = \frac{2}{\pi}\int_0^{\pi} e^x \sin nx\, dx \text{ for n being ever integer}$$

$$c_2 = \frac{-2n}{\pi(1+n^2)}\left(e^{\pi} + 1\right) \tag{36}$$

Hence the solution of generalized reaction diffusion problem with above specific boundary condition is obtained by replacing c_1 and c_2 by (35) and (36) in equation (30)

Example: 3

Let the initial value of function u (x,t) in the solution of generalized reaction diffusion problem is given by u (x, 0) = f (x) = x

The solution is given by (30) and

c_1 is obtained by

$$\left(\frac{1}{\frac{2R}{n\left(R+Dn^2\right)}\left(\frac{n}{2}\right)^p+c_1}\right)^{\frac{1}{p-1}}\frac{2}{\pi}\int_0^{\pi}\sin nx\,dx \text{ for n being odd integer}$$

$$=\frac{2}{\pi}\left(-x\frac{\cos nx}{n}+\frac{\sin nx}{n^2}\right)_0^{\pi}=\frac{2}{n}$$

Hence $$\left(\frac{1}{\frac{2R}{n\left(R+Dn^2\right)}\left(\frac{n}{2}\right)^p+c_1}\right)^{\frac{1}{p-1}}=\frac{2}{n} \qquad (37)$$

c_2 is obtained by

$$c_2=\frac{2}{\pi}\int_0^{\pi}\sin\ nx\ dx \text{ for n being even integer}$$

$$=\frac{2}{\pi}\left(-x\frac{\cos nx}{n}+\frac{\sin nx}{n^2}\right)_0^{\pi}=-\frac{2}{n}$$

$$c_2=\frac{-2}{n} \qquad (38)$$

Hence the solution of generalized reaction diffusion problem with above specific condition is obtained by replacing c_1 and c_2 by (37) and (38) respectively in equation (30).

Conclusion

Here we find the solution of generalized reaction diffusion model with blow up condition. As we all are aware that the change in the boundary conditions in partial differential equation leads the change in solution of partial differential equations. In this paper we also illustrate the example of different nature of function. By the same way the user from different field may use it with different boundary condition. The area is here, open for wide applications in various field of research.

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