Some Applications of rth order Extorial function in Generalized Difference Operator

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Abstract:

In this paper, using the nth order positive and negative extorial function, we derive higher order difference operator values of nth order positive and negative extorial function. Correct examples are inserted to explain the main theorems.

1. Introduction

The difference operator Δ , $\Delta u(k) = u(k+1) - u(k)$ where $\{u(k)\}$ sequence of numbers. The factorial polynomial is $k_l^{(s)} = \prod_{r=o}^{s-1}(k - rl)$. This extorial function verify the difference equation $\Delta_l^r\, u(k) = v(k)$.

Definition: 1.1

If $l \neq 0$ is any real and $u(k)$ is any real valued function, Then the difference operator on $u(k)$ defined as,

$\Delta_l u(k) = u(k+l) - u(k)$

Example : if $u(k) = e^{2k}$,

Then $\Delta_l u(k) = = e^{2(k+l)} - e^{2k}$

$\Delta_l^2 u(k) = = e^{2(k+2l)} - 2e^{2(k+l)} + e^{2k}$

$\Delta_l^3 u(k) = = e^{2(k+3l)} - 3e^{2(k+2l)} + 3e^{2(k+l)} - e^{2k}$

$\Delta_l^4 u(k) = e^{2(k+4l)} - 4e^{2(k+3l)} + 6e^{2(k+2l)} - 4e^{2(k+l)} + e^{2k}$

Definition: 1.2

The generalized polynomial factorial is,

$k_l^{(s)} = k(k-l)(k-2l)\ldots(k-(s-1)l)$

Lemma: 1.3

For fixed $k \in \mathbb{R}, l, r \in \mathbb{N}$, we have

(i) $\Delta_l\left(k_l^{(r)}\right) = rl\, k_l^{(r-1)}$

(ii) $\Delta_l\left(\frac{1}{k_l^{(r)}}\right) = \frac{-rl}{(k+l)_l^{(r+1)}}$

(iii) $\Delta_l^{-1}\left(k_l^{(r)}\right) = \frac{k_l^{(r+1)}}{l(r+1)} + C$

2. rth order positive extorial function

Definition: 2.1

If $-1 < l < 1, k\ is\ real,$ then r^{th} order extorial function is defined as $e_r(k_l) = 1 + \frac{k_l^{(r)}}{r!} + \frac{k_l^{(2r)}}{(2r)!} + \frac{k_l^{(3r)}}{(3r)!} + \cdots + \infty$ …..(1)

Lemma: 2.2

For $k \in \mathbb{R}$, and $l, r \in \mathbb{N}$,

i) $$e_r(-k_l) = \begin{cases} e_r\left(k_{(-l)}\right), & if\ r\ is\ even \\ 1 - \frac{k_{(-l)}^{(r)}}{r!} + \frac{k_{(-l)}^{(2r)}}{(2r)!} - \frac{k_{(-l)}^{(3r)}}{(3r)!} + \cdots, & if\ r\ is\ odd \end{cases}$$

ii) $e_r\left(-k_{(-l)}\right) = \begin{cases} e_r\left(k_{(l)}\right), & if\ r\ is\ even \\ 1 - \frac{k_l^{(r)}}{r!} + \frac{k_l^{(2r)}}{(2r)!} - \frac{k_l^{(3r)}}{(3r)!} + \cdots, & if\ r\ is\ odd \end{cases}$

Theorem 2.3

Let $k \in \mathbb{N}$ and $r, l \in \mathbb{N}$ and $tr \neq 1, r \neq s$. Then we have $\Delta_l^s\left(e_r(k_l)\right) = l^s \sum_{t=1}^{\infty} \frac{k_l^{(tr-s)}}{(tr-s)!}$

Proof:

We have $e_r(k_l) = 1 + \frac{k_l^{(r)}}{r!} + \frac{k_l^{(2r)}}{(2r)!} + \frac{k_l^{(3r)}}{(3r)!} + \cdots + \infty$

$$= \left[\sum_{t=0}^{\infty} \frac{k_l^{(tr)}}{(tr)!}\right]$$

$$\Delta_l(e_r(k_l)) = \Delta_l \left[\sum_{t=0}^{\infty} \frac{k_l^{(tr)}}{(tr)!}\right]$$

$$\Delta_l(e_r(k_l)) = l \sum_{t=1}^{\infty} \frac{k_l^{(tr-1)}}{(tr-1)!}$$

$$\Delta_l^2\left(e_r(k_l)\right) = \Delta_l\left(\Delta_l e_r(k_l)\right)$$

$$= \Delta_l \left[l \sum_{t=1}^{\infty} \frac{k_l^{(tr-1)}}{(tr-1)!}\right]$$

$$\Delta_l^2\left(e_r(k_l)\right) = l^2 \sum_{t=1}^{\infty} \frac{k_l^{(tr-2)}}{(tr-2)!}$$

Similarly,

$$\Delta_l^3\left(e_r(k_l)\right) = l^3 \sum_{t=1}^{\infty} \frac{k_l^{(tr-3)}}{(tr-3)!}$$

$$\Delta_l^4\left(e_r(k_l)\right) = l^4 \sum_{t=1}^{\infty} \frac{k_l^{(tr-4)}}{(tr-4)!}$$

In general,

$$\Delta_l^s\left(e_r(k_l)\right) = l^s \sum_{t=1}^{\infty} \frac{k_l^{(tr-s)}}{(tr-s)!} \qquad \ldots\ldots\ldots\ldots(2)$$

Example:2.5

Put $l = 4, r = 2, s = 3$. These values substitute in previous lemma.

L.H.S in (2) = $\Delta_4^2(e_3(k_4))$

$$= \Delta_4^2\left(1 + \frac{k_4^{(3)}}{3!} + \frac{k_4^{(6)}}{6!} + \frac{k_4^{(9)}}{9!} + \cdots + \infty\right)$$

$$= 4^2\left[\frac{k_4^{(1)}}{1!} + \frac{k_4^{(4)}}{4!} + \frac{k_4^{(7)}}{7!} + \cdots + \infty\right]$$

R.H.S in (2) $= l^n \sum_{t=1}^{\infty} \frac{k_l^{(tm-n)}}{(tm-n)!}$

$$= 4^2 \sum_{t=1}^{\infty} \frac{k_4^{(3t-2)}}{(3t-2)!}$$

$$= 4^2\left[\frac{k_4^{(1)}}{1!} + \frac{k_4^{(4)}}{4!} + \frac{k_4^{(7)}}{7!} + \cdots + \infty\right]$$

L.H.S = R.H.S

Lemma: 2.6

For any positive s and real $k, r \neq s$. We have $\Delta_l^{-s}\left(e_r(k_l)\right) = \frac{1}{l^s} \sum_{t=0}^{\infty} \frac{k_l^{(ts+r)}}{(ts+r)!}$

Proof:

We have $e_r(k_l) = \left[\sum_{t=0}^{\infty} \frac{k_l^{(tr)}}{(tr)!}\right]$

$$\Delta_l^{-1}\left(e_r(k_l)\right) = \Delta_l^{-1}\left[\sum_{t=0}^{\infty} \frac{k_l^{(tr)}}{(tr)!}\right]$$

$\Delta_l^{-1}\big(e_r(k_l)\big) = \frac{1}{l}\sum_{t=0}^{\infty}\frac{k_l^{(tr+1)}}{(tr+1)!}$

$$\Delta_l^{-2}\big(e_r(k_l)\big) = \Delta_l^{-1}\Big(\Delta_l^{-1}\big(e_r(k_l)\big)\Big)$$

$$= \Delta_l^{-1}\left[\frac{1}{l}\sum_{t=0}^{\infty}\frac{k_l^{(tr+1)}}{(tr+1)!}\right]$$

$$= \frac{1}{l}\left(\sum_{t=0}^{\infty}\frac{k_l^{(tr+2)}}{l(tr+2)!}\right)$$

$$= \frac{1}{l^2}\left(\sum_{t=0}^{\infty}\frac{k_l^{(tr+2)}}{(tr+2)!}\right)$$

$\Delta_l^{-2}\big(e_r(k_l)\big) = \frac{1}{l^2}\sum_{t=0}^{\infty}\frac{k_l^{(tr+2)}}{(tr+2)!}$

Similarly,

$\Delta_l^{-3}\big(e_r(k_l)\big) = \frac{1}{l^3}\sum_{t=0}^{\infty}\frac{k_l^{(tr+3)}}{(tr+3)!}$

$\Delta_l^{-4}\big(e_r(k_l)\big) = \frac{1}{l^4}\sum_{t=0}^{\infty}\frac{k_l^{(tr+4)}}{(tr+4)!}$

In general,

$\Delta_l^{-s}\big(e_r(k_l)\big) = \frac{1}{l^s}\sum_{t=0}^{\infty}\frac{k_l^{(tr+s)}}{(tr+s)!},\ r \neq s$

3. r^th order negative extorial function

Definition: 3.1

If $|l| < 1$ and k is any real ,the negative r^th order extorial function is

$e_{(-r)}(k_l) = 1 + \frac{1}{r!}\,\frac{1}{k_l^{(r)}} + \frac{1}{(2r)!}\,\frac{1}{k_l^{(2r)}} + \frac{1}{(3r)!}\,\frac{1}{k_l^{(3r)}} + \cdots + \infty$

Lemma: 3.2

For $k > 0, -1 < l < 1, s \neq r$ and $\frac{1}{k_l^{(r)}} = k_l^{(-r)}$, we have

$$\Delta_l^s\big(e_{(-r)}(k_l)\big) = (-1)^s\, l^s \sum_{p=1}^{\infty}\frac{(pr+1)(pr+2)\ldots(pr+(s-1))}{(pr-1)!\ (k+sl)_l^{(pr+s)}}$$

Proof:

We have $e_{(-r)}(k_l) = 1 + \frac{1}{r!}\,\frac{1}{k_l^{(r)}} + \frac{1}{(2r)!}\,\frac{1}{k_l^{(2r)}} + \frac{1}{(3r)!}\,\frac{1}{k_l^{(3r)}} + \cdots + \infty$

$=\sum_{p=0}^{\infty}\frac{1}{(pr)!k_l^{(pr)}}$

$$\Delta_l e_{(-r)}(k_l) = \Delta_l\left(\sum\nolimits_{p=0}^{\infty}\frac{1}{(pr)!\,k_l^{(tr)}}\right)$$

$$= \left[\sum\nolimits_{p=1}^{\infty}\frac{-lpn}{pr(pr-1)!\,(k+l)_l^{(pn+1)}}\right]$$

$$\Delta_l e_{(-r)}(k_l) = -l\sum_{p=1}^{\infty}\frac{1}{(pr-1)!\,(k+l)_l^{(pn+1)}}$$

$$\Delta_l^2 e_{(-r)}(k_l) = \Delta_l\left[-l\sum_{p=1}^{\infty}\frac{1}{(pr-1)!\,(k+l)_l^{(pr+1)}}\right]$$

$$\Delta_l^2 e_{(-r)}(k_l) = l^2\sum_{p=1}^{\infty}\frac{(pr+1)}{(pr-1)!\,(k+2l)_l^{(pr+2)}}$$

Simillarly $\Delta_l^3(e_{(-r)}(k_l)) = -l^3\sum_{p=1}^{\infty}\frac{(pr+1)(pr+2)}{(pr-1)!(k+3l)_l^{(pr+3)}}$

In general,

$$\Delta_l^s\big(e_{(-r)}(k_l)\big) = (-1)^s\, l^s\sum_{p=1}^{\infty}\frac{(pr+1)(pr+2)\ldots(pr+(s-1))}{(pr-1)!\ (k+sl)_l^{(pr+s)}}$$

Theorem: 3.3

For $-1 < l < 1, r \neq s$,k>0 and $\frac{1}{k_l^{(r)}} = k_l^{(-r)}$, we have

$$\Delta_l^{-s}\big(e_{(-r)}(k_l)\big) = \frac{1}{l^s}\left[\frac{k_l^{(s)}}{s!}+\sum_{p=1}^{\infty}\frac{1}{(pr)!\,(pr-1)(pr-2)\ldots(pr-s)(k-sl)_l^{(pr-s)}}\right]$$

Proof:

We have,

$$e_{(-r)}(k_l) = 1+\frac{1}{r!}\frac{1}{k_l^{(r)}}+\frac{1}{(2r)!}\frac{1}{k_l^{(2r)}}+\frac{1}{(3r)!}\frac{1}{k_l^{(3r)}}+\cdots+\infty$$

$$=\sum_{p=0}^{\infty}\frac{1}{(pr)!k_l^{(pr)}}$$

$$\Delta_l^{-1}(e_{(-r)}(k_l)) = \Delta_l^{-1}\sum_{p=0}^{\infty}\frac{1}{(pr)!k_l^{(pr)}}$$

$$\Delta_l^{-1}(e_{(-r)}(k_l)) = \frac{1}{l}\left[\frac{k_l^{(1)}}{1!}-\sum_{p=1}^{\infty}\frac{1}{(pr)!(pr-1)(k-l)_l^{(pr-1)}}\right]$$

$$\Delta_l^{-2}(e_{(-r)}(k_l)) = \Delta_l^{-1}\frac{1}{l}\left[\frac{k_l^{(1)}}{1!}-\sum_{p=1}^{\infty}\frac{1}{(pr)!\,(pr-1)(k-l)_l^{(pr-1)}}\right]$$

$$\Delta_l^{-2}(e_{(-r)}(k_l)) = \frac{1}{l^2}\left[\frac{k_l^{(2)}}{2!}+\sum_{p=1}^{\infty}\frac{1}{(pr)!\,(pr-1)(pr-2)(k-2l)_l^{(pr-2)}}\right]$$

Similarly,

$$\Delta_l^{-3}(e_{(-r)}(k_l)) = \frac{1}{l^3}\left[\frac{k_l^{(3)}}{3!}-\sum_{p=1}^{\infty}\frac{1}{(pr)!\,(pr-1)(pr-2)(pr-3)(k-3l)_l^{(pr-3)}}\right]$$

In general,

$$\Delta_l^{-s}\big(e_{(-r)}(k_l)\big) = \frac{1}{l^s}\left[\frac{k_l^{(s)}}{s!}+(-1)^s\sum_{p=1}^{\infty}\frac{1}{(pr)!\,(pr-1)(pr-2)\ldots(pr-s)(k-sl)_l^{(pr-s)}}\right]$$

Conclusion

The n order positive and negative extorial functions are explained the values of n^{th} order difference operator values. Here we solved the values for the inverse higher order difference operator.

Reference

[1] Manual. M, Adem Klman, G. Xavier, R. Pugalarasu and D.S. Dilip. “On the solutions of second order generalised difference equations”, Advances in Difference Equations, 2012.

[2] G. Britto Antony Xavier, S. John Borg, S. Jarald Pushparaj. “Extorial solutions for fractional and partial difference equations with applications”, AJP Publishing 2019.

[3] G. Britto Antony Xavier, S. John Borg, Sathinathan, “Extorial function and its properties in Discrete Calculus”, AJP Publishing 2020.

[4] M.Maria Susai Manuel,G.Britto Antony Xavier and E.Thandapani,Theory of Generalized difference operator and its Applications,For East Journal of Mathematical science20(2),(2006)

[5] G.Britto Antony Xavier,M.Meganathan,Fractional order alpha Laplace and Extorial transform by inverse difference operator,AIP Conference proceedings 2095,030016,2019

[6] G.Britto Antony Xavier ,S.Jerald Pushparaj Extorial solutions for fractional and partial difference equations with applications,AIP conference proceedings 2095,030004(2019)

[7] S.John Borg,G.Britto Antony XavierS.Jerald Pushparaj,Extorial solutions for Discrete and fractional difference equations of current flows in RL circuit ,Journal of applied science and computations 6(1),(2019)

[8] T.Sathinathan ,K.Vignesh G.Britto Antony Xavier ,Two dimensional second order Fibonacci summation formula with Extorial functions Journal of physics conference series 1597(1) 012006 (2020)

[9] Jarald Pushparaj Simon, T.Sathinathan,G.Britto Antony Xavier,Fibonacci triangular matrices and its application in mittag leffler fractional function using difference operators AIP Conference proceedings 2385,130054 (2022)

[10] P.Rajinikanth ,G.Britto Antony Xavier, Advanced Fibonacci sequence and its sum by second order Variable co-efficient difference operator Journal of computational mathematica 2456-8686, 5(2) ,2021:92-101, https://doi.org/10.26524/cm112

[11] T.Sathinathan ,Jarald Pushparaj Simon ,G.Britto Antony Xavier,Extra sub exponential and extorial functions in differential and difference equations, AIP Conference proceedings 2385,130055(2022)

[12] M.Sumathy,P.Venkata Mohan Reddy, Maria Susai Manuel Qualitative property of third order non linear neutral distributed deley -Generalized difference Equation, Mathematical problems in Engineering volume 2021, https://doi.org/10.1155/2021/2875613