UNIQUENESS AND WEIGHTED SET SHARING OF MEROMORPHIC FUNCTIONS CONCERNING TWO DIFFERENTIAL POLYNOMIALS

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Abstract

In this article, we use the concept of weighted set sharing and study uniqueness of two differential polynomials which shares a set of small functions. Results obtained here generalizes the results of Zhang and Lii, Banerjee and Majumder, Charak and Lal and D.C Pramanik.

Keywords: Uniqueness Meromorphic Function Small Function Weighted Set Sharing Differential Polynomial.

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1 Introduction and Definitions

In this article, a non-constant meromorphic functions denoted by f in the complex plane $\mathbb{C}$. Throughout the paper, the standard notations of Nevanlinna theory are adapted as explained in [1, 2, 3, 4]. It is convenient to let E denote a set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function f, we denote $T(r, f)$ the Nevanlinna characteristic of f and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$, as $r \to \infty$, $r \notin E$. We represent the small function by $\gamma(z)$ with respect to $f(z)$ if $T(r, \gamma(z)) = S(r, f)$. The collection of all small functions with respect to f is denoted by $S(f)$. Clearly $\mathbb{C} \cup \{\infty\} \in S(f)$ is a field over the set of complex number. If $a \in \mathbb{C} \cup \{\infty\}$, then

$$\delta(a, f) = 1 - \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r, f)}$$

and

$$\Theta(a, f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}\left(r, \frac{1}{f-a}\right)}{T(r, f)}$$

are called the deficiency and ramification index of a function f respectively.

Definition 1.1 Let β be non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote $E_\beta(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \le \beta$ and $\beta + 1$ times if $m > \beta$. If $E_\beta(a; f) = E_\beta(a; g)$ we say that f, g share the value a with weight β,

Clearly, if f, g share (a, β), then f, g share (a, p) for any integer p, $0 \le p \le \beta$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ and (a, ∞) respectively.

We say that two function f and g share the set S CM (IM) if $E_f(S) = E_g(S)$ $(\overline{E}_f(S) = \overline{E}_f(S))$ where the set

$S \subset \mathbb{C} \cup \{\infty\}$.

Definition 1.2 [5]. An expression of the form

$$P[f] = \sum_{i=1}^{t} a_i(z) f^{n_{i0}} (f')^{n_{i1}} \ldots\ldots (f^{(k)})^{n_{it}} \tag{1.1}$$

is called as differential polynomial in f of degree $\overline{d}(P)$, lower degree $\underline{d}(P)$ and weight Γ_P, where $n_{i0}, n_{i1} \ldots\ldots n_{ik}$ are non-negative integers and $a_i(z)$ are meromorphic functions satisfying $T(r, a_i) = S(r, f)$ and

$$\overline{d}(P) = max\{\sum_{j=0}^{k} n_{ij} \ ; \ 1 \le i \le t\}$$

$$\underline{d}(P) = min\{\sum_{j=0}^{k} n_{ij} \ ; \ 1 \le i \le t\}$$

$$\Gamma_P = max\{\sum_{j=0}^{k} (j+1) n_{ij} \ ; \ 1 \le i \le t\}.$$

If $\overline{d}P = \underline{d}P = n$ (say), then we say that $P[f]$ is called a homogeneous differential polynomial in f of degree n. Otherwise it is called as non-homogeneous differential polynomial.

From last two decades researchers are interested in proving uniqueness results between entire(or meromorphic) function and their derivatives. One can refer Rubel and Yang [11] for f and f' share α, β CM where $\alpha \ne \beta$ and Mues and Steinmertz [1979] for f and f' share α, β IM where $\alpha \ne \beta$. In 2020, Banerjee and Mazumder [17] answered the open question posed by Zhang and L$\ddot{u}$ [16] using the concept of weighted sharing method sharing a small function `a' for the functions f' and $(f^m)^{(k)}$. Later on Charak and Lal [18] used the concept for proving uniqueness between polynomial of degree $n \ge 1$ with $p(0) = 0$ and a non-constant differential polynomial. In 2019, D.C. Pramanik [19] used the concept weighted set sharing to prove uniqueness results of the $p(f)$ and $P[f]$.

Motivated by these results, we used the concept of weighted set sharing to study the uniqueness results of two non-homogeneous differential polynomials and prove the following results.

Theorem 1.11 Let f and g be two non-constant meromorphic functions. Let $a(z) \in S(f) \cap S(g)$ and $a \ne 0, \infty$. Suppose that $H[f]$ and $H[g]$ are two differential polynomials as defined in (1.1) which are non-constant. If $H[f]$ and $H[g]$ share (S_m, l) with one of the following conditions hold:

(i) $l \ge 2$ and

$$\begin{aligned} & min\{(3 + mQ + 2m\Gamma - 2m\underline{d}H)\Theta(\infty, f) + mQ\Theta(0, f) + 2m\delta_{2+\Gamma-\underline{d}(H)}(0, f^{\underline{d}(H)}) \\ & ,(3 + mQ + 2m\Gamma - 2m\underline{d}(H))\Theta(\infty, g) + mQ\Theta(0, g) + 2m\delta_{2+\Gamma-\underline{d}(H)}(0, g^{\underline{d}(H)})\} \\ & \le (2mQ + 2m\Gamma + 4m\overline{d}H - 7m\underline{d}(H) + 3 + 2m) \end{aligned} \tag{1.2}$$

(ii) $l = 1$ and

$$min\{\left(\frac{2mQ+5m\Gamma-5m\underline{d}(H)+7}{2}\right)\Theta(\infty,f)+mQ\Theta(0,f)$$
$$+\frac{m}{2}\delta_{1+\Gamma-\underline{d}(H)}(0,f^{\underline{d}(H)})+2m\delta_{2+\Gamma-\underline{d}(H)}(0,f^{\underline{d}(H)}),$$
$$\left(\frac{2mQ+5m\Gamma-5m\underline{d}(H)+7}{2}\right)\Theta(\infty,g)+mQ\Theta(0,g)$$
$$+\frac{m}{2}\delta_{1+\Gamma-\underline{d}(H)}(0,g^{\underline{d}(H)})+2m\delta_{2+\Gamma-\underline{d}(H)}(0,g^{\underline{d}(H)})\}$$
$$\leq\left(\frac{4mQ+5m\Gamma+10m\bar{d}H-17m\underline{d}(H)+7+5m}{2}\right) \quad (1.3)$$

(iii) $l=0$ and

$$min\{(mQ+5m\Gamma-5m\underline{d}(H)+6)\Theta(\infty,f)+mQ\Theta(0,f)$$
$$+2m\delta_{2+\Gamma-\underline{d}(H)}(0,f^{\underline{d}(H)})+3m\delta_{1+\Gamma-\underline{d}(H)}(0,f^{\underline{d}(H)}),$$
$$(mQ+5m\Gamma-5m\underline{d}(H)+6)\Theta(\infty,g)+mQ\Theta(0,g)$$
$$+2m\delta_{2+\Gamma-\underline{d}(H)}(0,g^{\underline{d}(H)})+3m\delta_{1+\Gamma-\underline{d}(H)}(0,g^{\underline{d}(H)})\}$$
$$\leq(2mQ+5m\Gamma+10\bar{d}H-16\underline{d}(H)+6+5m) \quad (1.4)$$

Then either $H[f]\equiv H[g]$ or $H[f]H[g]=1$

Lemmas

In this section we use the following lemmas which are needed to prove our theorem. **Lemma 2.1 [6]:** Let f be a non-constant meromorphic function and $P[f]$ be a differential polynomial of f. Then

$$m\left(r,\frac{P[f]}{f^{\bar{d}P}}\right)\leq(\bar{d}P-\underline{d}P)m\left(r,\frac{1}{f}\right)+S(r,f)$$

$$N\left(r,\frac{P[f]}{f^{\bar{d}P}}\right)\leq(\bar{d}P-\underline{d}P)N\left(r,\frac{1}{f}\right)+Q\left[\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)\right]+S(r,f)$$

$$N\left(r,\frac{1}{P[f]}\right)\leq Q\overline{N}(r,f)+(\bar{d}P-\underline{d}P)m\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f^{\bar{d}(P)}}\right)+S(r,f).$$

Where $Q=max\,1\leq i\leq m\,\{n_{i0}+n_{i1}+2n_{i2}+\ldots+kn_{ik}\}$

Lemma 2.2 [7]: Let f and g be two non-constant meromorphic functions. If f and g share $(l,0)$, then

$$\overline{N}_L\left(r,\frac{1}{f-1}\right)\leq\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+S(r).$$

Lemma 2.3 [8]: Let f and g be two non-constant meromorphic functions. If f and g share (l,l), then

$$2\overline{N}_L\left(r,\frac{1}{f-1}\right)+2\overline{N}_L\left(r,\frac{1}{g-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{f-1}\right)-\overline{N}_{f>2}\left(r,\frac{1}{g-1}\right)$$
$$\leq N\left(r,\frac{1}{g-1}\right)-\overline{N}\left(r,\frac{1}{g-1}\right).$$

Lemma 2.4 [9]: Let j and p be two positive integers satisfying $j \geq p+1$ and $\Gamma > (k+1)\underline{d}(H)-(P+1)$, then for a differential polynomial $H[f]$

$$\begin{aligned} N_p(r,0;H[f]) &\leq N_{p+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+(\Gamma-\underline{d}(H))\overline{N}(r,\infty;f) \\ &+(\overline{d}(H)-\underline{d}(H))\left[m\left(r,\frac{1}{f}\right)+T(r,f)\right]+S(r,f). \end{aligned}$$

Lemma 2.5 [10]: For a differential polynomial $H[f]$, we have

$$\underline{d}(H)T(r,f)-Q\left[\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)\right]+S(r,f)$$
$$\leq T(r,H[f])\leq Q\overline{N}(r,f)+\overline{d}(H)T(r,f)+S(r,f).$$

Proof of Theorem 1.11:

Let

$$F_1=\frac{H[f]}{a} \quad \text{and} \quad G_1=\frac{H[g]}{a}$$

Set

$$F=(F_1)^m, \quad G=(G_1)^m$$

Then F and G share $(1,l)$ with the possible exception of the zeros and poles of $a(z)$

Also we have

$$\overline{N}(r,F)=\overline{N}(r,f)+S(r,f)$$
$$\overline{N}(r,G)=\overline{N}(r,g)+S(r,g)$$

We define

$$\psi=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right) \tag{3.1}$$

Suppose that $\psi \not\equiv 0$. Then $m(r,\psi)=S(r,f)$. By Second Fundamental Theorem of Nevanlinna we have

$$\begin{aligned} T(r,F)+T(r,G) &\leq \overline{N}(r,f)+\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G}\right) \\ &+\overline{N}\left(r,\frac{1}{G-1}\right)-\overline{N}_0\left(r,\frac{1}{F'}\right)-\overline{N}_0\left(r,\frac{1}{G'}\right)+S(r,f) \end{aligned} \tag{3.2}$$

Case 1: For $l \geq 1$, if z_0 is a common simple 1-point of F and G. Then substituting their Taylor series at z_0 in $\psi(z)$. We see that z_0 is a zero of $\psi(z)$. Then we get

$$\begin{aligned} N_E^{1)}\left(r,\frac{1}{F-1}\right) &\le N\left(r,\frac{1}{\psi}\right)+S(r,f) \\ &\le T(r,\psi)+S(r,f) \\ &= N(r,\psi)+S(r,f) \\ &\le \overline{N}(r,F)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right) \\ &+\overline{N}_L\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f) \end{aligned} \tag{3.3}$$

Here $N_0\left(r,\frac{1}{F'}\right)$ is the counting function which only counts those points such that $F'=0$ but $F(F-1)\neq 0$ since F and G share 1 IM, it is easy to see that

$$\begin{aligned} \overline{N}\left(r,\frac{1}{F-1}\right) &= N_E^{1)}\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{G-1}\right) \\ &+N_E^{(2}\left(r,\frac{1}{F-1}\right) \\ &= \overline{N}\left(r,\frac{1}{G-1}\right) \end{aligned}$$

Substituting for $N_E^{1)}\left(r,\frac{1}{F-1}\right)$, using (3.3) we get

$$\begin{aligned} \overline{N}\left(r,\frac{1}{F-1}\right) &\le \overline{N}(r,F)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+2\overline{N}_L\left(r,\frac{1}{F-1}\right) \\ &+2\overline{N}_L\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+N_E^{(2}\left(r,\frac{1}{F-1}\right)+S(r,f) \end{aligned} \tag{3.4}$$

Subcase 1.1: If $l=1$, we have

$$\begin{aligned} \overline{N}_L\left(r,\frac{1}{F-1}\right) &\le \frac{1}{2}N\left(r,\frac{1}{F'}\mid F\neq 0\right) \\ &\le \frac{1}{2}\overline{N}(r,F)+\frac{1}{2}\overline{N}(r,\frac{1}{F}), \end{aligned} \tag{3.5}$$

where $N\left(r,\frac{1}{F'}\mid F\neq 0\right)$ denotes the zeros of F', which are not the zeros of F. Now from (2.5) and (3.5) we get

$$\begin{aligned} 2\overline{N}_L\left(r,\frac{1}{F-1}\right) &+2\overline{N}_L\left(r,\frac{1}{G-1}\right)+N_E^{(2}\left(r,\frac{1}{F-1}\right)+N\left(r,\frac{1}{G-1}\right) \\ &\le N\left(r,\frac{1}{G-1}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)+S(r,f) \\ &\le N\left(r,\frac{1}{G-1}\right)+\frac{1}{2}\overline{N}(r,F)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)+S(r,f) \end{aligned} \tag{3.6}$$

From (3.4) and (3.6) we have

$$\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\le \overline{N}(r,F)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)$$
$$+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\frac{1}{2}\overline{N}(r,F)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)+N\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F'}\right)$$
$$+N_0\left(r,\frac{1}{G'}\right)+S(r,f) \tag{3.7}$$

From (2.3), (3.2) and (3.7) we get

$$T(r,F)+T(r,G)\le \overline{N}(r,f)+\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,f)$$
$$+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\frac{1}{2}\overline{N}(r,F)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)+T(r,G)+N_0\left(r,\frac{1}{F'}\right)$$
$$+N_0\left(r,\frac{1}{G'}\right)-N_0\left(r,\frac{1}{F'}\right)-N_0\left(r,\frac{1}{G'}\right)+S(r,f)$$

Here $\overline{N}(r,g)=\overline{N}(r,f)$, $\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)\le N_2\left(r,\frac{1}{F}\right)$

$$\therefore T(r,F)\quad \le \frac{7}{2}\overline{N}(r,f)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+S(r,f)$$
$$T(r,F_1^m)\quad \le \frac{7}{2}\overline{N}(r,f)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F_1^m}\right)+N_2\left(r,\frac{1}{F_1^m}\right)+N_2\left(r,\frac{1}{G_1^m}\right)+S(r,f) \tag{3.8}$$

Using Lemma (2.4), Lemma (2.5) and (3.8)

$$m\left\{\underline{d}(H)T(r,f)-Q\left[\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)\right]\right\}$$
$$\le\left(\frac{m(3\Gamma-3\underline{d}(H))+7}{2}\right)\overline{N}(r,f)+m(\Gamma-\underline{d}(H))\overline{N}(r,g)+3m(\overline{d}(H)$$
$$-\underline{d}(H))T(r,f)+2m(\overline{d}(H)-\underline{d}(H))T(r,g)+\frac{m}{2}N_{2+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})$$
$$+mN_{2+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+mN_{2+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})+S(r,f)+S(r,g)$$

$$m(4\underline{d}(H)-3\overline{d}(H))T(r,f)$$
$$\le\left(\frac{m(3\Gamma-3\underline{d}(H)+2Q)+7}{2}\right)\overline{N}(r,f)+m(\Gamma-\underline{d}(H))\overline{N}(r,g)$$
$$+2m(\overline{d}(H)-\underline{d}(H))T(r,g)+mQ\overline{N}\left(r,\frac{1}{f}\right)+\frac{m}{2}N_{1+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})$$
$$+mN_{2+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+mN_{2+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})+S(r,f)+S(r,g) \tag{3.9}$$

Similarly

$$
\begin{aligned}
&m(4\underline{d}(H)-3\overline{d}(H))T(r,g)\\
&\leq\left(\frac{m(3\Gamma-3\underline{d}(H)+2Q)+7}{2}\right)\overline{N}(r,g)+m(\Gamma-\underline{d}(H))\overline{N}(r,f)\\
&+2m(\overline{d}(H)-\underline{d}(H))T(r,f)+mQ\overline{N}\left(r,\frac{1}{g}\right)\\
&+\frac{m}{2}N_{1+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})+mN_{2+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})\\
&+mN_{2+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+S(r,f)+S(r,g)
\end{aligned} \tag{3.10}
$$

From (3.9) and (3.10)

$$
\begin{aligned}
&m(6\underline{d}(H)-5\overline{d}(H))(T(r,f)+T(r,g))\leq\left(\frac{m(5\Gamma-5\underline{d}(H)+2Q)+7}{2}\right)\\
&+mQ\left[\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)\right]\\
&+\frac{m}{2}N_{1+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+\frac{m}{2}N_{1+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})\\
&+2mN_{2+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+2mN_{2+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})+S(r,f)+S(r,g)
\end{aligned}
$$

Now consider

$$
\begin{aligned}
&m(6\underline{d}(H)-5\overline{d}(H))T(r,f)\leq\left(\frac{m(5\Gamma-5\underline{d}(H)+2Q)+7}{2}\right)\overline{N}(r,f)\\
&+mQ\overline{N}\left(r,\frac{1}{f}\right)+\frac{m}{2}N_{1+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+2mN_{2+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})
\end{aligned}
$$

$$
\begin{aligned}
&\left(\frac{2mQ+5m\Gamma-5m\underline{d}(H)+7}{2}\right)\Theta(\infty,f)+mQ\Theta(0,f)\\
&+\frac{m}{2}\delta_{1+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+2m\delta_{2+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})\\
&\leq\left(\frac{4mQ+5m\Gamma-17m\underline{H}+10m\overline{d}(H)+5m+7}{2}\right)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&\left(\frac{2mQ+5m\Gamma-5m\underline{d}(H)+7}{2}\right)\Theta(\infty,g)+mQ\Theta(0,g)\\
&+\frac{m}{2}\delta_{1+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})+2m\delta_{2+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})\\
&\leq\left(\frac{4mQ+5m\Gamma-17m\underline{d}(H)+10m\overline{d}(H)+5m+7}{2}\right)
\end{aligned}
$$

$$min\{\left(\frac{2mQ+5m\Gamma-5m\underline{d}(H)+7}{2}\right)\Theta(\infty,f)+mQ\Theta(0,f)$$

$$+\frac{m}{2}\delta_{1+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+2m\delta_{2+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)}),$$

$$\left(\frac{2mQ+5m\Gamma-5m\underline{d}(H)+7}{2}\right)\Theta(\infty,g)+mQ\Theta(0,g)$$

$$+\frac{m}{2}\delta_{1+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})+2m\delta_{2+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})\}$$

$$\leq\left(\frac{4mQ+5m\Gamma-17m\underline{d}(H)+10m\overline{d}(H)+5m+7}{2}\right)$$

Subcase 1.2: If $l\geq 2$

In this case, we have

$$2\overline{N}_L\left(r,\frac{1}{F-1}\right)+2\overline{N}_L\left(r,\frac{1}{G-1}\right)+N_E^{(2}\left(r,\frac{1}{F-1}\right)+N\left(r,\frac{1}{G-1}\right)$$

$$\leq N\left(r,\frac{1}{G-1}\right)+S(r,f)$$

$$\leq T(r,G)+S(r,F)+S(r,G)$$

Therefore from (3.4), we obtain

$$\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\leq\overline{N}(r,F)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)$$

$$+N\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f)$$

$$\leq\overline{N}(r,F)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+T(r,G)+N_0\left(r,\frac{1}{F'}\right)$$

$$+N_0\left(r,\frac{1}{G'}\right)+S(r,f)$$

(3.11)

From lemma (2.3), (3.2) and (3.11), we have

$$T(r,F)+T(r,G)\leq\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,f)$$

$$+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+T(r,G)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)$$

$$-N_0\left(r,\frac{1}{F'}\right)-N_0\left(r,\frac{1}{G'}\right)+S(r,f)+S(r,g)$$

$$T(r,F)\leq N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+3\overline{N}(r,f)+S(r,f)$$

$$(3m\underline{d}(H)-2m\overline{d}(H))T(r,f)-2m(\overline{d}(H)-\underline{d}(H))T(r,g)$$
$$\leq (3+mQ+m(\Gamma-\underline{d}(H)))\overline{N}(r,f)+mQ\overline{N}\left(r,\frac{1}{f}\right)+m(\Gamma-\underline{d}(H))\overline{N}(r,g)$$
$$+mN_{2+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+mN_{2+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})+S(r,f)+S(r,g) \quad (3.12)$$

Similarly

$$(3m\underline{d}(H)-2m\overline{d}(H))T(r,g)-2m(\overline{d}(H)-\underline{d}(H))T(r,f)$$
$$\leq (3+mQ+m(\Gamma-\underline{d}(H)))\overline{N}(r,g)+mQ\overline{N}\left(r,\frac{1}{g}\right)+m(\Gamma-\underline{d}(H))\overline{N}(r,g)$$
$$+mN_{2+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+mN_{2+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})+S(r,f)+S(r,g) \quad (3.13)$$

On adding (3.12) and (3.13) we get

$$(3m\underline{d}(H)-2m\overline{d}(H))[T(r,f)+T(r,g)]-2m(\overline{d}(H)-\underline{d}(H))[T(r,f)+T(r,g)]$$
$$\leq (3+mQ+m(\Gamma-\underline{d}(H)))[\overline{N}(r,f)+\overline{N}(r,g)]+mQ\left[\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)\right]$$
$$+m(\Gamma-\underline{d}(H))[\overline{N}(r,f)+\overline{N}(r,g)]+2mN_{2+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})$$
$$+2mN_{2+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})+S(r,f)+S(r,g)$$

$$(5m\underline{d}(H)-4m\overline{d}(H))[T(r,f)+T(r,g)]$$
$$\leq (3+mQ+2m(\Gamma-\underline{d}(H)))[\overline{N}(r,f)+\overline{N}(r,g)]+mQ\left[\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)\right]$$
$$+2mN_{2+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+2mN_{2+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})+S(r,f)+S(r,g)$$

Consider

$$(5m\underline{d}(H)-4m\overline{d}(H))T(r,f)\leq (3+mQ+2m\Gamma-2m\underline{d}(H))\overline{N}(r,f)+mQ\overline{N}\left(r,\frac{1}{f}\right)$$
$$+2mN_{2+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+S(r,f)$$

$$(3m+mQ+2m\Gamma-2m\underline{d}(H))\Theta(\infty,f)+mQ\Theta(0,f)+2m\delta_{2+\Gamma-\underline{d}(H)}(0;f^{\underline{d}(H)})$$
$$\leq 2mQ+2m\Gamma-7m\underline{d}(H)+4m\overline{d}(H)+3+2m$$

Similarly

$$(3m+mQ+2m\Gamma-2m\underline{d}(H))\Theta(\infty,g)+mQ\Theta(0,g)+2m\delta_{2+\Gamma-\underline{d}(H)}(0;g^{\underline{d}(H)})$$
$$\leq 2mQ+2m\Gamma-7m\underline{d}(H)+4m\overline{d}(H)+3+2m$$

Therefore

$$min[(3m+mQ+2m\Gamma-2m\underline{d}(H))\Theta(\infty,f)+mQ\Theta(0,f)+2m\delta_{2+\Gamma-\underline{d}(H)}(0;f^{\underline{d}(H)}),$$
$$(3m+mQ+2m\Gamma-2m\underline{d}(H))\Theta(\infty,g)+mQ\Theta(0,g)+2m\delta_{2+\Gamma-\underline{d}(H)}(0;g^{\underline{d}(H)})]$$
$$\leq 2mQ+2m\Gamma-7m\underline{d}(H)+4m\overline{d}(H)+3+2m$$

Subcase 1.3: If $l=0$

In this case, we have

$$N_E^{1)}\left(r,\frac{1}{F-1}\right) \quad +2N_E^{(2}\left(r,\frac{1}{G-1}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)+2\overline{N}_L\left(r,\frac{1}{G-1}\right)$$
$$\leq N\left(r,\frac{1}{G-1}\right)+S(r,F)$$
$$\leq T(r,G)+S(r,F)+S(r,G) \tag{3.14}$$

Also

$$\overline{N}_L\left(r,\frac{1}{F-1}\right) \quad \leq N\left(r,\frac{1}{F-1}\right)-\overline{N}\left(r,\frac{1}{F-1}\right)$$
$$\leq N\left(r,\frac{F}{F'}\right)\leq N\left(r,\frac{F'}{F}\right)+S(r,F)$$
$$\leq \overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,F\right)+S(r,F) \tag{3.15}$$

Similarly

$$\overline{N}_L\left(r,\frac{1}{G-1}\right) \quad \leq \overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,G)+S(r,G) \tag{3.16}$$

Combining (3.2), (3.14), (3.15) and (3.16) we get

$$T(r,F)\leq N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+6\overline{N}(r,f)+2\overline{N}\left(r,\frac{1}{F}\right)+N_1\left(r,\frac{1}{G}\right)+S(r,f)$$

By Lemma 2.4 and Lemma 2.5, we get

$$m\left\{\underline{d}(H)T(r,f)-Q\left[\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)\right]\right\}$$
$$\leq (3m\Gamma-3m\underline{d}(H)+6)\overline{N}(r,f)+2m(\Gamma-\underline{d}(H))\overline{N}(r,g)+6m(\overline{d}(H)-\underline{d}(H))T(r,f)$$
$$+4m(\overline{d}(H)-\underline{d}(H))T(r,g)+mN_{2+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+mN_{2+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})$$
$$+2mN_{1+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+mN_{1+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})+S(r,f)+S(r,g)$$

$$(7m\underline{d}(H)-6m\overline{d}(H))T(r,f)$$
$$\leq (mQ+3m\Gamma-3m\underline{d}(H)+6)\overline{N}(r,f)+2m(\Gamma-\underline{d}(H)))\overline{N}(r,g)$$
$$+4m(\overline{d}(H)-\underline{d}(H))T(r,g)+mQ\overline{N}\left(r,\frac{1}{f}\right)+mN_{2+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})$$
$$+mN_{2+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})+2mN_{1+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})$$
$$+mN_{1+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})+S(r,f)+S(r,g) \tag{3.17}$$

Similarly

$$
\begin{aligned}
&(7m\underline{d}(H)-6m\overline{d}(H))T(r,g)\\
&\leq (mQ+3m\Gamma-3m\underline{d}(H)+6)\overline{N}(r,g)+2m(\Gamma-\underline{d}(H))\overline{N}(r,f)\\
&+4m(\overline{d}(H)-\underline{d}(H))T(r,f)+mQ\overline{N}\left(r,\frac{1}{g}\right)+mN_{2+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})\\
&+mN_{2+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+2mN_{1+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})\\
&+mN_{1+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+S(r,f)+S(r,g)
\end{aligned} \tag{3.18}
$$

On adding (3.17) and (3.18), we get

$$
\begin{aligned}
&(11m\underline{d}(H)-10\overline{d}(H))[T(r,f)+T(r,g)]\\
&\leq (mQ+5m\Gamma-5m\underline{d}(H)+6)[\overline{N}(r,f)+\overline{N}(r,g)]+mQ\left[\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)\right]\\
&+2mN_{2+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+2mN_{2+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})+3mN_{1+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})\\
&+3mN_{1+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})+S(r,f)+S(r,g)
\end{aligned}
$$

Now Consider

$$
\begin{aligned}
&(11m\underline{d}(H)-10\overline{d}(H))T(r,f)\\
&\leq (mQ+5m\Gamma-5m\underline{d}(H)+6)\overline{N}(r,f)+mQ\overline{N}\left(r,\frac{1}{f}\right)\\
&+2mN_{2+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+3mN_{1+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+S(r,f)
\end{aligned}
$$

$$
\begin{aligned}
&(mQ+5m\Gamma-5m\underline{d}(H)+6)\Theta(\infty,f)+mQ\Theta(0,f)+2m\delta_{2+\Gamma-\underline{d}(H)}(0,f^{\underline{d}(H)})\\
&+3m\delta_{1+\Gamma-\underline{d}(H)}(0,f^{\underline{d}(H)})\leq 2mQ+5m\Gamma-16\underline{d}(H)+10\overline{d}(H)+6+5m
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&(mQ+5m\Gamma-5m\underline{d}(H)+6)\Theta(\infty,g)+mQ\Theta(0,g)+2m\delta_{2+\Gamma-\underline{d}(H)}(0,g^{\underline{d}(H)})\\
&+3m\delta_{1+\Gamma-\underline{d}(H)}(0,g^{\underline{d}(H)})\leq 2mQ+5m\Gamma-16\underline{d}(H)+10\overline{d}(H)+6+5m
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&min\{(mQ+5m\Gamma-5m\underline{d}(H)+6)\Theta(\infty,f)+mQ\Theta(0,f)+2m\delta_{2+\Gamma-\underline{d}(H)}(0,f^{\underline{d}(H)})\\
&+3m\delta_{1+\Gamma-\underline{d}(H)}(0,f^{\underline{d}(H)}),(mQ+5m\Gamma-5m\underline{d}(H)+6)\Theta(\infty,g)+mQ\Theta(0,g)\\
&+2m\delta_{2+\Gamma-\underline{d}(H)}(0,g^{\underline{d}(H)})+3m\delta_{1+\Gamma-\underline{d}(H)}(0,g^{\underline{d}(H)})\}\\
&\leq (2mQ+5m\Gamma+10\overline{d}H-16\underline{d}(H)+6+5m)
\end{aligned}
$$

This proves the claim and $\psi \equiv 0$

i.e

$$
\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right)
$$

Integrating, we obtain

$$
\frac{1}{F-1}=\frac{C}{G-1}+D \tag{3.19}
$$

where $c \neq 0$ and D are constants. Now the following three cases arise

Case (i): If $D \neq 0,-1$

Rearranging (3.19) as

$$\frac{G-1}{C}=\frac{F-1}{D+1-DF}$$

We have

$$\overline{N}(r,G)=\overline{N}\left(r,\frac{1}{F-\frac{D+1}{D}}\right)+S(r,f)$$

By Second Fundamental Theorem of Nevanlinna and Lemma 2.4, we have

$$m\left\{\underline{d}(H)T(r,f)-Q\left[\overline{N}(r,f)+\overline{N}\left(\frac{1}{f}\right)\right]\right\}+S(r,f)\leq T(r,H[f])$$

$$\begin{aligned}&m\left\{\underline{d}(H)T(r,f)-Q\left[\overline{N}(r,f)+\overline{N}\left(\frac{1}{f}\right)\right]\right\}\\&\leq\overline{N}(r,H[f])+\overline{N}\left(\frac{1}{H[f]}\right)+\overline{N}(r,H[g])+S(r,f)\\&\leq\overline{N}(r,f)+N_{1+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+(\overline{d}(H)-\underline{d}(H))2T(r,f)\\&+(\Gamma-\underline{d}(H))\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+S(r,f)\end{aligned}$$

$$\begin{aligned}&m\left\{\underline{d}(H)T(r,f)-Q\overline{N}(r,f)-Q\overline{N}\left(\frac{1}{f}\right)\right\}\leq(1+\Gamma-\underline{d}(H))\overline{N}(r,f)\\&+\overline{N}(r,g)+2(\overline{d}(H)-\underline{d}(H))T(r,f)+N_{1+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+S(r,f)\end{aligned}$$

$$\begin{aligned}&\left((m+2)\underline{d}(H)-2\overline{d}(H)\right)T(r,f)\leq(1+mQ+\Gamma-\underline{d}(H))\overline{N}(r,f)+\overline{N}(r,g)\\&+mQ\overline{N}\left(r,\frac{1}{f}\right)+N_{1+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+S(r,f)\end{aligned}\tag{3.20}$$

Similarly

$$\begin{aligned}&\left((m+2)\underline{d}(H)-2\overline{d}(H)\right)T(r,g)\leq(1+mQ+\Gamma-\underline{d}(H))\overline{N}(r,g)+\overline{N}(r,f)\\&+mQ\overline{N}\left(r,\frac{1}{g}\right)+N_{1+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})+S(r,f)\end{aligned}\tag{3.21}$$

On adding (3.20) and (3.21), we get

$$\begin{aligned}&[(m+2)\underline{d}(H)-2\overline{d}(H)][T(r,f)+T(r,g)]\\&\leq(2+mQ+\Gamma-\underline{d}(H))[\overline{N}(r,f)+\overline{N}(r,g)]+mQ\left[\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)\right]\\&+N_{1+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+N_{1+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})+S(r,f)+S(r,g)\end{aligned}$$

Now consider

$$[(m+2)\underline{d}(H)-2\overline{d}(H)]T(r,f)\le(2+mQ+\Gamma-\underline{d}(H))\overline{N}(r,f)$$
$$+N_{1+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+mQ\overline{N}\left(r,\frac{1}{f}\right)+S(r,f)$$

We get

$$(2+mQ+\Gamma-\underline{d}(H))\Theta(\infty,f)+mQ\Theta(0,f)+\delta_{1+\Gamma-\underline{d}(H)}(0;f^{\underline{d}(H)})$$
$$\le 2mQ+\Gamma-(m+3)\underline{d}(H)$$
$$+2\overline{d}(H)+3$$

Similarly

$$(2+mQ+\Gamma-\underline{d}(H))\Theta(\infty,g)+mQ\Theta(0,g)+\delta_{1+\Gamma-\underline{d}(H)}(0;g^{\underline{d}(H)})$$
$$\le 2mQ+\Gamma-(m+3)\underline{d}(H)+2\overline{d}(H)+3$$

Therefore

$$min\{(2+mQ+\Gamma-\underline{d}(H))\Theta(\infty,f)+mQ\Theta(0,f)+\delta_{1+\Gamma-\underline{d}(H)}(0;f^{\underline{d}(H)})$$
$$,(2+mQ+\Gamma-\underline{d}(H))\Theta(\infty,g)+mQ\Theta(0,g)+\delta_{1+\Gamma-\underline{d}(H)}(0;g^{\underline{d}(H)})$$
$$+2\overline{d}(H)+3\}\le 2mQ+\Gamma-(m+3)\underline{d}(H)+2\overline{d}(H)+3$$

Which contradicts (1.2), (1.3) and (1.4).

Case (ii): If D=0, then we have

$$G=CF-(C-1) \tag{3.22}$$

Therefore, if $c\neq 1$, then

$$\overline{N}\left(r,\frac{1}{G}\right)=\overline{N}\left(r,\frac{1}{F-\frac{C-1}{C}}\right)$$

By Nevanlinna’s Second Fundamental Theorem and Lemma 2.4, we get

$$m\left\{\underline{d}(H)T(r,f)-Q\left[\overline{N}(r,f)+\overline{N}\left(\frac{1}{f}\right)\right]\right\}\le T(r,H[f])$$
$$\le\overline{N}(r,H[f])+\overline{N}\left(r,\frac{1}{H[f]}\right)+\overline{N}\left(r,\frac{1}{H[g]}\right)+S(r,g)$$
$$\le(1+\Gamma-\underline{d}(H))\overline{N}(r,f)+2(\overline{d}(H)-\underline{d}(H))T(r,f)+2(\overline{d}(H)-\underline{d}(H))T(r,g)$$
$$+(\Gamma-\underline{d}(H))\overline{N}(r,g)+N_{1+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+N_{1+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})+S(r,f)$$

$$\left((m+2)\underline{d}(H)-2\overline{d}(H)\right)T(r,f)\le(1+mQ+\Gamma-\underline{d}(H))\overline{N}(r,f)+mQ\overline{N}\left(r,\frac{1}{f}\right)$$
$$+(\Gamma-\underline{d}(H))\overline{N}(r,g)+2(\overline{d}(H)-\underline{d}(H))T(r,g)+N_{1+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})$$
$$+N_{1+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})+S(r,f) \tag{3.23}$$

Similarly

$$\left((m+2)\underline{d}(H)-2\overline{d}(H)\right)T(r,g)\le(1+mQ+\Gamma-\underline{d}(H))\overline{N}(r,g)+mQ\overline{N}\left(r,\frac{1}{g}\right)$$
$$+(\Gamma-\underline{d}(H))\overline{N}(r,f)+2(\overline{d}(H)-\underline{d}(H))T(r,f)+N_{1+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})$$
$$+N_{1+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+S(r,g) \quad (3.24)$$

On adding (3.23) and (3.24)

$$[(m+2)\underline{d}(H)-2\overline{d}(H)][T(r,f)+T(r,g)]\le(1+mQ+\Gamma-\underline{d}(H))[\overline{N}(r,f)+\overline{N}(r,g)]$$
$$+mQ\left[\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)\right]+(\Gamma-\underline{d}(H))[\overline{N}(r,f)+\overline{N}(r,g)]$$
$$+2(\overline{d}(H)-\underline{d}(H))[T(r,f)+T(r,g)]+2N_{1+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})$$
$$+2N_{1+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})+S(r,f)+S(r,g)$$

$$[(m+4)\underline{d}(H)-4\overline{d}(H)][T(r,f)+T(r,g)]$$
$$\le(1+mQ+2\Gamma-2\underline{d}(H))[\overline{N}(r,f)+\overline{N}(r,g)]+mQ\left[\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)\right]$$
$$+2N_{1+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+2N_{1+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})+S(r,f)+S(r,g)$$

Now Consider

$$[(m+4)\underline{d}(H)-4\overline{d}(H)]T(r,f)\le(1+mQ+2\Gamma-2\underline{d}(H))\overline{N}(r,g)$$
$$+mQ\overline{N}\left(r,\frac{1}{f}\right)+2N_{1+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})+S(r,f)$$

Then we get

$$(1+mQ+2\Gamma-2\underline{d}(H))\Theta(\infty,f)+mQ\Theta(0,f)+2\delta_{1+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)})$$
$$\le 2mQ+2\Gamma-(6+m)\underline{d}(H)+4\overline{d}(H)+3$$

Similarly

$$(1+mQ+2\Gamma-2\underline{d}(H))\Theta(\infty,g)+mQ\Theta(0,g)+2\delta_{1+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})$$
$$\le 2mQ+2\Gamma-(6+m)\underline{d}(H)+4\overline{d}(H)+3$$

$$min\{(1+mQ+2\Gamma-2\underline{d}(H))\Theta(\infty,f)+mQ\Theta(0,f)+2\delta_{1+\Gamma-\underline{d}(H)}(r,0;f^{\underline{d}(H)}),$$
$$(1+mQ+2\Gamma-2\underline{d}(H))\Theta(\infty,g)+mQ\Theta(0,g)+2\delta_{1+\Gamma-\underline{d}(H)}(r,0;g^{\underline{d}(H)})\}$$
$$2mQ+2\Gamma-(6+m)\underline{d}(H)+4\overline{d}(H)+3$$

Which contradicts to (1.2), (1.3) and (1.4). Thus $C=1$ and hence, in this case we have $F=G$.

$$\therefore H[f]=H[g]$$

Case (iii): If $D\equiv-1$, then we have

$$\frac{1}{F-1}\equiv\frac{C}{G-1}-1 \quad (3.25)$$

Therefore if $C \neq -1$, then $\overline{N}\left(r, \frac{1}{G}\right) = \overline{N}\left(r, \frac{1}{F - \frac{C}{C+1}}\right)$ By using Nevanlinna's Second Fundamental Theorem and Lemma (2.4), we get the conclusion which is the same as case (ii)

$$min\{(1+mQ+2\Gamma-2\underline{d}(H))\Theta(\infty, f)+mQ\Theta(0, f)+2\delta_{1+\Gamma-\underline{d}(H)}(r,0; f^{\underline{d}(H)}),$$
$$(1+mQ+2\Gamma-2\underline{d}(H))\Theta(\infty, g)+mQ\Theta(0, g)+2\delta_{1+\Gamma-\underline{d}(H)}(r,0; g^{\underline{d}(H)})\}$$
$$\leq 2mQ+2\Gamma-(6+m)\underline{d}(H)+4\overline{d}(H)+3$$

which contradicts to (1.2), (1.3) and (1.4). Therefore $C = -1$ and so in this case from (3.16), we obtain

$$FG \equiv 1$$

$$\therefore\ H[f]H[g] \equiv 1$$

Therefore, we get either $H[f] \equiv H[g]$ or $H[f]H[g] \equiv 1$. Hence the proof of the Theorem 1.1

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