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Codes Over Group $Q_{4n}\times C_t$

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Abstract: In this article, we study the group codes that are one sided ideals in a semisimple group algebra $F(Q_{4n} \times C_t)$ where $Q_{4n}$is generalized quaternion group of order 4n and C_t is the cyclic group of order t. We also find the minimum distance and dimension of the group codes generated by the idempotents.

Keywords: Group algebra, group codes, characters, linear and non idempotents.

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1. Introduction and Preliminaries

A group code is defined as an ideal in the group algebra FG where G is finite group and F is a field. If $char(F) \nmid O(G)$ then FG is semisimple and is direct some of some minimal ideals. Every ideal is generated by some idempotent. In this paper, we give explicit expressions for the generating idempotents of the codes in the group algebra of direct product of generalized quaternion group Q_{4n} and cyclic group C_t and also find the minimum distance and dimension.

Let E = $\{e_i\}_{i=1}^{s}$ be the set of all idempotents in FG . If I is any ideal generated by $\{e_j\}_{j=1}^{t} \subseteq$ E and μ=E\\$\{e_j\}_{j=1}^{t}$ then $I = \{ u \in FG : ue = 0 \ \forall e \in \mu \}$. Now we denote I by I_μ.

The weight of any element $u = \sum_{g\in G} \alpha_g g \in FG$ is equal to number of non zero components in u and is denoted by *wt(u)*. Minimum distance of the group code I_μ is defined as $d(I_\mu)$= min$\{wt(u): 0 \neq u \in I_{\{e_i\}}\}$. The length n of a group code I_μ is defined to be order of G. If I_μ has dimension k and minimum distance d then $I_\mu$is called an (n,k,d) group code.

Remark 1.1 (see [1])We know $FG= \left(\bigoplus_{ei\in E_L} FG_{e_i}\right) \oplus \left(\bigoplus_{ej\in E_N} FG_{e_j}\right)$ where E_L is the set consisting of all linear idempotents in FG and E_N is the set consisting of all nonlinear idempotents in FG and $FG_{e_i}$is minimal ideal generated by e_i. Thus E= $E_L \cup E_N$. Note that if $e_i \in E_L$, then $dim(FG_{e_i}) = 1$; and if $e_j \in E_N$, then $dim\left(FG_{e_j}\right) = m^2$ where $m = \chi_k(1)$ where χ_k is the k[th] non linear character of G. Therefore if $\mu = \mu_L \cup \mu_N$ where $\mu_L \subseteq E_L$ and $\mu_N \subseteq E_N$, then $dim(I_\mu)= dim(FG) - |\mu_L| dim(FG_{e_i}) - |\mu_N| dim(FG_{e_j})$ where $dim(FG) = |G|$.

The product of two characters of G is again a character of G. Let χ be the character of V and ψ be the character of W then the character of $V \times W$ is $\chi \times \psi$, where $(\chi \times \psi)(g,h) = \chi(g)\psi(h) \quad (g \in V, h \in W)$.

Theorem 1.2 [4] Let $\chi_1, \dots, \chi_s$ be the distinct irreducible characters of G and let $\psi_1, \dots, \psi_t$ be the distinct irreducible characters of H.Then $G \times H$ has precisely st distinct irreducible characters, and these are $\chi_i \times \psi_j \ (1 \le i \le s, 1 \le j \le t)$.

Theorem 1.3 Q_{4n} has $n+3$ conjugacy classes [4] which are given by

$$\{1\}, \{a^n\}, \{a^r, a^{-r}\}(1 \le r \le n-1), \{a^{2j}b : 0 \le j \le n-1\}, \{a^{2j+1}b : 0 \le j \le n-1\}.$$

Remark: Corresponding to $n+3$ conjugacy classes there are $n+3$ irreducible characters in Q_{4n}.

2. Idempotents in the group algebra $F(Q_{4n} \times C_t)$

Consider generalized quaternion group $Q_{4n} = < a, b: a^{2n} = 1, a^n = b^2, ab = ba^{-1} >$ of order $4n$ and $C_t = < g > = \{1, g, g^2, \dots\dots. g^{t-1}\}$ is cyclic group of order t. Define: $x_1 = 1,\ x_2 = a,\ x_3 = a^2, \dots, x_n = a^{n-1}$, $x_{n+1} = a^n, \dots, x_{2n} = a^{2n-1}$, $x_{2n+1} = b, x_{2n+2} = ab, \ . \ . \ ., \quad x_{4n} = a^{2n-1}b, \quad y_1 = 1, y_2 = g, y_3 = g^2, \dots, y_t = g^{t-1}$.

Let $G = Q_{4n} \times C_t = \{(x_i, y_j)\}\ 1 \le i \le 4n, 1 \le j \le t$ be the direct product of Q_{4n} and C_t.

2.1 Explicit expressions for the $(n+3)t$ irreducible idempotents in the group algebra $F(Q_{4n} \times C_t)$ when n is an even number.

The group Q_{4n} has $n+3$ irreducible characters and C_t has t irreducible characters. By using the Theo.1.2, $Q_{4n} \times C_t$ has $(n+3)t$ irreducible characters. The characters of $Q_{4n} \times C_t$ are given by

$\chi_1(x) = 1$ for all $x \in Q_{4n} \times C_t$

$\chi_2(a^r, g^l) = 1, \chi_2(a^r b, g^l) = -1$ where $0 \le r \le 2\text{n} - 1$

$\chi_3(1, g^l) = 1, \chi_3(a^r, g^l) = (-1)^r\ (1 \le r \le \text{n}),\ \chi_3(b, g^l) = 1, \chi_3(ab, g^l) = -1$

$\chi_4(1, g^l) = 1, \chi_4(a^r, g^l) = (-1)^r\ (1 \le r \le \text{n}),\ \chi_4(b, g^l) = -1, \chi_4(ab, g^l) = 1$

$\varphi_k(a^r, g^l) = (\alpha^k)^l$, $\varphi_k(a^r b, g^l) = (\alpha^k)^l$ where $0 \le r \le 2\text{n} - 1$

$\phi_k(a^r, g^l) = (\alpha^k)^l$, $\phi_k(a^r b, g^l) = -(\alpha^k)^l$ where $0 \le r \le 2\text{n} - 1$

$\psi_k(1, g^l) = (\alpha^k)^l$, $\psi_k(a^r, g^l) = (-1)^r(\alpha^k)^l$ $(1 \le r \le \text{n})$, $\psi_k(b, g^l) = (\alpha^k)^l$, $\psi_k(ab, g^l) = -(\alpha^k)^l$

$\Phi_k(1, g^l) = (\alpha^k)^l$, $\Phi_k(a^r, g^l) = (-1)^r(\alpha^k)^l$ $(1 \le r \le \text{n})$, $\Phi_k(b, g^l) = -(\alpha^k)^l$, $\Phi_k(ab, g^l) = (\alpha^k)^l$

$\Upsilon_j(1, g^l) = 2$, $\Upsilon_j(a^n, g^l) = 2(-1)^j$, $\Upsilon_j(a^r, g^l) = 2\cos\frac{\pi r j}{n}$ $(1 \le r \le \text{n} - 1)$

$\Upsilon_j(a^s b, g^l) = 0$ where $0 \le s \le 2\text{n} - 1$

$\xi_{jk}(1, g^l) = 2(\alpha^k)^l$, $\xi_{jk}(a^n, g^l) = 2(-1)^j(\alpha^k)^l$,

$\xi_{jk}(a^r, g^l) = 2\cos\frac{\pi r j}{n}(\alpha^k)^l$ $(1 \le r \le \text{n} - 1)$, $\xi_{jk}(a^s b, g^l) = 0$ $(0 \le s \le 2\text{n} - 1)$,

where $0 \le l \le t - 1$, $1 \le j \le n-1$, $k = 1, 2, \dots, t-1$ and α is t^{th} root of unity.

The characters $\chi_1, \chi_2, \chi_3, \chi_4, \varphi_k, \phi_k, \psi_k, \Phi_k$ of degree one so these are linear and Υ_j, ξ_{jk} are non linear. Therefore the group algebra $F(Q_{4n} \times C_t)$ has $(n+3)t$ irreducible idempotents.

We define some notations

$\overline{c_1} = \sum_{i=1}^{2n}(x_i, y_1)$

$\overline{c_2} = \sum_{i=1}^{2n}(x_i, y_2)$

$\overline{c_3} = \sum_{i=1}^{2n}(x_i, y_3), \dots.,\ \overline{c_t} = \sum_{i=1}^{2n}(x_i, y_t)$

$\overline{c_{t+1}} = \sum_{\substack{i=1 \\ i \text{ is odd}}}^{2n-1}(x_{2n+i}, y_1)$

$\overline{c_{t+2}} = \sum_{\substack{i=2 \\ i \text{ is even}}}^{2n}(x_{2n+i}, y_1)$

$\overline{c_{t+3}} = \sum_{\substack{i=1 \\ i \text{ is odd}}}^{2n-1}(x_{2n+i}, y_2)$

$\overline{c_{t+4}} = \sum_{\substack{i=2 \\ i \text{ is even}}}^{2n}(x_{2n+i}, y_2), \dots, \overline{c_{3t-1}} = \sum_{\substack{i=1 \\ i \text{ is odd}}}^{2n-1}(x_{2n+i}, y_t)$

$\overline{c_{3t}} = \sum^{2n}_{\substack{i=2 \\ i \text{ is even}}} (x_{2n+i}, y_t)$

The idempotents of FG are given by using $e = \frac{1}{|G|}\sum_{g\in G}\chi(g^{-1})g$ (see[4, Pro.14.10])

Group $Q_{4n} \times C_t$ has $4t$ irreducible linear characters and corresponding linear idempotents of FG are given by

$e_1 = \frac{1}{4nt} \sum_{\substack{1\le i\le 4n \\ 1\le j\le t}}(x_i, y_j)$

$e_2 = \frac{1}{4nt}[\sum_{i=1}^{t}\overline{C_i} - \sum_{j=1}^{2t}\overline{C_{t+j}}]$

$e_3 = \frac{1}{4nt}[\sum_{j=1}^{t}(x_1, y_j) + (-1)^r\sum_{\substack{1\le r\le 2n-1 \\ 1\le j\le t}}(x_{r+1}, y_j) - \sum^{2n-1}_{\substack{i=1 \\ i \text{ is odd} \\ 1\le j\le t}}(x_{2n+i}, y_j) + \sum^{2n}_{\substack{i=2 \\ i \text{ is even} \\ 1\le j\le t}}(x_{2n+i}, y_j)]$

$e_4 = \frac{1}{4nt}[\sum_{j=1}^{t}(x_1, y_j) + (-1)^r\sum_{\substack{1\le r\le 2n-1 \\ 1\le j\le t}}(x_{r+1}, y_j) + \sum^{2n-1}_{\substack{i=1 \\ i \text{ is odd} \\ 1\le j\le t}}(x_{2n+i}, y_j) - \sum^{2n}_{\substack{i=2 \\ i \text{ is even} \\ 1\le j\le t}}(x_{2n+i}, y_j)]$

$\upsilon_k = \frac{1}{4nt}[\overline{c_1} + (\alpha^k)^{t-1}\overline{c_2} + \ldots + \alpha^k\,\overline{c_t} + \overline{c_{t+1}} + \overline{c_{t+2}} + (\alpha^k)^{t-1}(\overline{c_{t+3}} + \overline{c_{t+4}}) + \ldots + \alpha^k(\overline{c_{3t-1}} + \overline{c_{3t}})]$

$\eta_k = \frac{1}{4nt}[\overline{c_1} + (\alpha^k)^{t-1}\overline{c_2} + \ldots + \alpha^k\,\overline{c_t} - \overline{c_{t+1}} - \overline{c_{t+2}} - (\alpha^k)^{t-1}(\overline{c_{t+3}} + \overline{c_{t+4}}) - \ldots - \alpha^k(\overline{c_{3t-1}} + \overline{c_{3t}})]$

$\zeta_k = \frac{1}{4nt}[(x_1,y_1) + (\alpha^k)^{t-1}(x_1,y_2) + \ldots + \alpha^k(x_1,y_t) + \sum_{r=1}^{2n-1}(-1)^r(x_{r+1}, y_1) +$
$(\alpha^k)^{t-1}\sum_{r=1}^{2n-1}(-1)^r(x_{r+1}, y_2) + \ldots + \alpha^k\sum_{r=1}^{2n-1}(-1)^r(x_{r+1}, y_t) - \overline{c_{t+1}} + \overline{c_{t+2}} + \ldots + \alpha^k(-\overline{c_{3t-1}} + \overline{c_{3t}})]$

$\delta_k = \frac{1}{4nt}[(x_1,y_1) + (\alpha^k)^{t-1}(x_1,y_2) + \ldots + \alpha^k(x_1,y_t) + \sum_{r=1}^{2n-1}(-1)^r(x_{r+1}, y_1) +$
$(\alpha^k)^{t-1}\sum_{r=1}^{2n-1}(-1)^r(x_{r+1}, y_2) + \ldots + \alpha^k\sum_{r=1}^{2n-1}(-1)^r(x_{r+1}, y_t) + \overline{c_{t+1}} - \overline{c_{t+2}} + \ldots + \alpha^k(\overline{c_{3t-1}} - \overline{c_{3t}})]$

G has $(n-1)t$ non linear characters and corresponding non linear idempotents of FG are given by

$\vartheta_j = \frac{1}{2nt}[2\sum_{1\le j\le t}(x_1, y_j) + 2(-1)^j\sum_{1\le l\le t}(x_{n+1}, y_l) + 2\sum_{r=1}^{2n-1}\cos\frac{\pi jr}{n}\{(a^r,1) + (a^r,g) + \ldots + (a^r,g^{t-1})\}]$

$\gamma_{jk} = \frac{1}{2nt}[2(1,1) + 2(\alpha^k)^{t-1}(1,g) + \ldots + 2\alpha^k(1,g^{t-1}) + 2(-1)^j\{(a^n,1) + (\alpha^k)^{t-1}(a^n,g) + \ldots + \alpha^k(a^n,g^{t-1})\} +$
$2\sum_{r=1}^{n-1}\cos\frac{\pi jr}{n}\{(a^r,1) + (a^{-r},1)\} + 2(\alpha^k)^{t-1}\sum_{r=1}^{n-1}\cos\frac{\pi jr}{n}\{(a^r,g) + (a^{-r},g)\} + \ldots + 2\alpha^k$
$\sum_{r=1}^{n-1}\cos\frac{\pi jr}{n}\{(a^r, g^{t-1}) + (a^{-r}, g^{t-1})\}]$

where $1\le j\le n-1$, $k=1,2,\ldots,t-1$ and α is t^{th} root of unity i.e. $\alpha^t = 1$.

2.2 Explicit expressions for the $(n+3)t$ irreducible idempotents in the group algebra $F(Q_{4n}\times C_t)$ when n is an odd number.

When n is odd, the group Q_{4n} has $n+3$ irreducible characters and C_t has t irreducible characters so $Q_{4n}\times C_t$ has $(n+3)t$ irreducible characters and correspondingly there are $(n+3)t$ irreducible idempotents in the group algebra $F(Q_{4n}\times C_t)$.

The linear irreducible characters of $Q_{4n}\times C_t$ are given by

$\chi_1(x) = 1$ for all $x \in Q_{4n}\times C_t$

$\chi_2(a^r, g^l) = 1, \chi_2(a^r b, g^l) = -1$ where $0 \le r \le 2n-1$

$\chi_3(1, g^l) = 1, \chi_3(a^r, g^l) = (-1)^r$ $(1 \le r \le n)$, $\chi_3(b, g^l) = i, \chi_3(ab, g^l) = -i$

$\chi_4(1, g^l) = 1$, $\chi_4(a^r, g^l) = (-1)^r$ $(1 \le r \le n)$, $\chi_4(b, g^l) = -i$, $\chi_4(ab, g^l) = i$

$\varphi_k(a^r, g^l) = (\alpha^k)^l$, $\varphi_k(a^r b, g^l) = (\alpha^k)^l$ where $0 \le r \le 2n-1$

$\phi_k(a^r, g^l) = (\alpha^k)^l$, $\phi_k(a^r b, g^l) = -(\alpha^k)^l$ where $0 \le r \le 2n-1$

$\psi_k(1, g^l) = (\alpha^k)^l$, $\psi_k(a^r, g^l) = (-1)^r(\alpha^k)^l$ $(1 \le r \le n)$, $\psi_k(b, g^l) = i(\alpha^k)^l$, $\psi_k(ab, g^l) = -i(\alpha^k)^l$

$\Phi_k(1,g^l)=(\alpha^k)^l$, $\Phi_k(a^r,g^l)=(-1)^r(\alpha^k)^l$ $(1\le r\le n)$, $\Phi_k(b,g^l)=-i(\alpha^k)^l$, $\Phi_k(ab,g^l)=i(\alpha^k)^l$.

Corresponding to $4t$ irreducible linear characters the linear idempotents of FG are given by

$$e_1=\frac{1}{4nt}\sum_{\substack{1\le i\le 4n\\ 1\le j\le t}}(x_i,y_j)$$

$$e_2=\frac{1}{4nt}\left[\sum_{i=1}^{t}\overline{C_l}-\sum_{j=1}^{2t}\overline{C_{t+j}}\right]$$

$$e_3=\frac{1}{4nt}\left[\sum_{j=1}^{t}(x_1,y_j)+\sum_{\substack{1\le r\le 2n-1\\ 1\le j\le t}}(-1)^r(x_{r+1},y_j)-i\sum_{\substack{l=1\\ l\ is\ odd\\ 1\le j\le t}}^{2n-1}(x_{2n+l},y_j)+i\sum_{\substack{l=2\\ l\ is\ even\\ 1\le j\le t}}^{2n}(x_{2n+l},y_j)\right]$$

$$e_4=\frac{1}{4nt}\left[\sum_{j=1}^{t}(x_1,y_j)+\sum_{\substack{1\le r\le 2n-1\\ 1\le j\le t}}(-1)^r(x_{r+1},y_j)+i\sum_{\substack{l=1\\ l\ is\ odd\\ 1\le j\le t}}^{2n-1}(x_{2n+l},y_j)-i\sum_{\substack{l=2\\ l\ is\ even\\ 1\le j\le t}}^{2n}(x_{2n+l},y_j)\right]$$

$$\vartheta_k=\frac{1}{4nt}\left[\overline{c_1}+(\alpha^k)^{t-1}\overline{c_2}+\ldots+\alpha^k\,\overline{c_t}+\overline{c_{t+1}}+\overline{c_{t+2}}+(\alpha^k)^{t-1}(\overline{c_{t+3}}+\overline{c_{t+4}})+\ldots+\alpha^k(\overline{c_{3t-1}}+\overline{c_{3t}})\right]$$

$$\eta_k=\frac{1}{4nt}\left[\overline{c_1}+(\alpha^k)^{t-1}\overline{c_2}+\ldots+\alpha^k\,\overline{c_t}-\overline{c_{t+1}}-\overline{c_{t+2}}-(\alpha^k)^{t-1}(\overline{c_{t+3}}+\overline{c_{t+4}})-\ldots-\alpha^k(\overline{c_{3t-1}}+\overline{c_{3t}})\right]$$

$$\zeta_k=\frac{1}{4nt}\Big[(x_1,y_1)+(\alpha^k)^{t-1}(x_1,y_2)+\ldots+\alpha^k(x_1,y_t)+\sum_{r=1}^{2n-1}(-1)^r(x_{r+1},y_1)+$$
$$(\alpha^k)^{t-1}\sum_{r=1}^{2n-1}(-1)^r(x_{r+1},y_2)+\ldots+\alpha^k\sum_{r=1}^{2n-1}(-1)^r(x_{r+1},y_t)-i\,\overline{c_{t+1}}+i\,\overline{c_{t+2}}+\ldots+\alpha^k i(-\overline{c_{3t-1}}+\overline{c_{3t}})\Big]$$

$$\delta_k=\frac{1}{4nt}\Big[(x_1,y_1)+(\alpha^k)^{t-1}(x_1,y_2)+\ldots+\alpha^k(x_1,y_t)+\sum_{r=1}^{2n-1}(-1)^r(x_{r+1},y_1)+$$
$$(\alpha^k)^{t-1}\sum_{r=1}^{2n-1}(-1)^r(x_{r+1},y_2)+\ldots+\alpha^k\sum_{r=1}^{2n-1}(-1)^r(x_{r+1},y_t)+i\,\overline{c_{t+1}}-i\,\overline{c_{t+2}}+\ldots+\alpha^k i(\overline{c_{3t-1}}-\overline{c_{3t}})\Big]$$

where i is 4^{th} root of unity i.e. $i^4=1$ and α is t^{th} root of unity i.e. $\alpha^t=1$.

Remark: The non linear idempotents of FG are identical in both cases 2.1 and 2.2.

3. Minimum distance and dimension of codes

Definition 3.1 When n is even number, the set of all irreducible idempotents of $F(Q_{4n}\times C_t)$ is E=$\{e_1,e_2,e_3,e_4,\upsilon_k,\zeta_k,\delta_k,\eta_k,\vartheta_j,\gamma_{jk},(1\le j\le n-1),(k=1,2,\ldots,t-1)\}$. If $\mu\subseteq E$ define $I_\mu=\{u\in F(Q_{4n}\times C_t): ue=0\ \ \forall e\in\mu\}$ is a group code generated by the set of idempotents $\{E\setminus\mu\}$. e.g. $I_{\{\upsilon_k,\eta_k,\zeta_k\}}=\{u\in F(Q_{4n}\times C_t): u\upsilon_k=u\eta_k=u\zeta_k=0\}$ is group code generated by the idempotents $\{e_1,e_2,e_3,e_4,\delta_k,\vartheta_j,\gamma_{jk}\}$.

Theorem 3.2 If $\mu\subseteq E_L$ then I_μ is $(4nt, 4nt-|\mu|, 2)$ group code.

Proof. When n is an even number

Let $u=\sum_{i=1}^{2n}\lambda_i(x_i,y_1)+\sum_{i=1}^{2n}\lambda_{2n+i}(x_i,y_2)+\ldots+\sum_{i=1}^{2n}\lambda_{2n(t-1)+i}(x_i,y_t)+\sum_{i=1}^{2n}\lambda_{2nt+i}(x_{2n+i},y_1)+\ldots+\sum_{i=1}^{2n}\lambda_{(2t-1)2n+i}(x_{2n+i},y_t)$ be any element of $F(Q_{4n}\times C_t)$ then

$$ue_1=\left(\sum_{i=1}^{4nt}\lambda_i\right)e_1 \qquad (3.1)$$

$$ue_2=\left(\sum_{i=1}^{2nt}\lambda_i-\sum_{i=2nt+1}^{4nt}\lambda_i\right)e_2 \qquad (3.2)$$

$$ue_3=\left(\sum_{i=1}^{2nt}\lambda_i(-1)^{i+1}+\sum_{i=2nt+1}^{4nt}\lambda_i(-1)^{i+1}\right)e_3 \qquad (3.3)$$

$$ue_4=\left(\sum_{i=1}^{2nt}\lambda_i(-1)^{i+1}+\sum_{i=2nt+1}^{4nt}\lambda_i(-1)^{i}\right)e_4 \qquad (3.4)$$

$$u\upsilon_k=\Big(\sum_{i=1}^{2n}\lambda_i+\alpha^k\sum_{i=2n+1}^{4n}\lambda_i+(\alpha^k)^2\sum_{i=4n+1}^{6n}\lambda_i+\ldots+(\alpha^k)^{t-1}\sum_{i=(t-1)2n+1}^{2nt}\lambda_i+\sum_{i=2nt+1}^{2n(t+1)}\lambda_i+\alpha^k\sum_{i=(t+1)2n+1}^{(t+2)2n}\lambda_i+\ldots+(\alpha^k)^{t-1}\sum_{i=(2t-1)2n+1}^{4nt}\lambda_i\Big)\vartheta_k \qquad (3.5)$$

$$u\eta_k=\Big(\sum_{i=1}^{2n}\lambda_i+\alpha^k\sum_{i=2n+1}^{4n}\lambda_i+(\alpha^k)^2\sum_{i=4n+1}^{6n}\lambda_i+\ldots+(\alpha^k)^{t-1}\sum_{i=(t-1)2n+1}^{2nt}\lambda_i-\sum_{i=2nt+1}^{2n(t+1)}\lambda_i-\alpha^k\sum_{i=(t+1)2n+1}^{(t+2)2n}\lambda_i-\ldots-(\alpha^k)^{t-1}\sum_{i=(2t-1)2n+1}^{4nt}\lambda_i\Big)\eta_k \qquad (3.6)$$

$u\zeta_k$ = ($\sum_{i=1}^{2n} \lambda_i$ $(-1)^{i+1}$ + $\alpha^k \sum_{i=2n+1}^{4n} \lambda_i(-1)^{i+1}$ + $(\alpha^k)^2 \sum_{i=4n+1}^{6n} \lambda_i(-1)^{i+1}$ + …..+ $(\alpha^k)^{t-1} \sum_{i=(t-1)2n+1}^{2nt} \lambda_i(-1)^{i+1}$ + $\sum_{i=2nt+1}^{2n(t+1)} \lambda_i(-1)^{i+1}$ + . . . + $(\alpha^k)^{t-1} \sum_{i=(2t-1)2n+1}^{4nt} \lambda_i(-1)^{i+1}$) ζ_k (3.7)

$u\delta_k$ = ($\sum_{i=1}^{2n} \lambda_i$ $(-1)^{i+1}$ + $\alpha^k \sum_{i=2n+1}^{4n} \lambda_i(-1)^{i+1}$ + $(\alpha^k)^2 \sum_{i=4n+1}^{6n} \lambda_i(-1)^{i+1}$ + …..+ $(\alpha^k)^{t-1} \sum_{i=(t-1)2n+1}^{2nt} \lambda_i(-1)^{i+1}$ + $\sum_{i=2nt+1}^{2n(t+1)} \lambda_i(-1)^{i}$ + . . . + $(\alpha^k)^{t-1} \sum_{i=(2t-1)2n+1}^{4nt} \lambda_i(-1)^{i}$)$\delta_k$ (3.8)

When n is odd number

$ue_3 = (\sum_{r=1}^{2nt} \lambda_r(-1)^{r+1} + i\sum_{r=2nt+1}^{4nt} \lambda_r(-1)^{r+1})\ e_3$ (3.9)

$ue_4 = (\sum_{r=1}^{2nt} \lambda_r(-1)^{r+1} + i\sum_{r=2nt+1}^{4nt} \lambda_r(-1)^{r})\ e_4$ (3.10)

$u\zeta_k$ = ($\sum_{r=1}^{2n} \lambda_r$ $(-1)^{r+1}$ + $\alpha^k \sum_{r=2n+1}^{4n} \lambda_r(-1)^{r+1}$ + $(\alpha^k)^2 \sum_{r=4n+1}^{6n} \lambda_r(-1)^{r+1}$ + …..+ $(\alpha^k)^{t-1} \sum_{r=(t-1)2n+1}^{2nt} \lambda_r(-1)^{r+1}$ + $i\sum_{r=2nt+1}^{2n(t+1)} \lambda_r(-1)^{r+1}$ + . . . + $(\alpha^k)^{t-1}$ $i\sum_{r=(2t-1)2n+1}^{4nt} \lambda_r(-1)^{r+1}$)$\zeta_k$

(3.11)

$u\delta_k$ = ($\sum_{r=1}^{2n} \lambda_r$ $(-1)^{r+1}$ + $\alpha^k \sum_{r=2n+1}^{4n} \lambda_r(-1)^{r+1}$ + $(\alpha^k)^2 \sum_{r=4n+1}^{6n} \lambda_r(-1)^{r+1}$ + …..+ $(\alpha^k)^{t-1} \sum_{r=(t-1)2n+1}^{2nt} \lambda_r(-1)^{r+1}$ + $i\sum_{r=2nt+1}^{2n(t+1)} \lambda_r(-1)^{r}$ + . . . + $(\alpha^k)^{t-1}$ $i\sum_{r=(2t-1)2n+1}^{4nt} \lambda_r(-1)^{r}$)$\delta_k$ (3.12)

Let $\mu = \{e_1\}$, now we find the minimum distance and dimension of $I_{\{e_1\}}$. Let $u = \lambda g \in F(Q_{4n} \times C_t)$ be any code word with $wt(u) = 1$. By equation (3.1) $ue_1 = \lambda e_1 \neq 0$, implies that $u = \lambda g \notin I_{\{e_1\}}$ and so $d(I_{\{e_1\}}) \geq 2$. If we take $u = \lambda_1(x_1, y_1) + \lambda_2(x_2, y_1) \in F(Q_{4n} \times C_t)$ then $ue_1 = 0$ if and only if $\lambda_1 = -\lambda_2$. So $u \in I_{\{e_1\}}$ hence $d(I_{\{e_1\}})$=2. From remark 1.1, we have $dim(I_{\{e_1\}})$ = 4nt−1. Hence $I_{\{e_1\}}$ is $(4nt, 4nt-1, 2)$ group code. Similarly we can prove for other values of μ.

4. Codes over $F(Q_{12} \times C_3)$

In this section we completely describe the codes over $F(Q_{12} \times C_3)$ when $n = 3$ and $t = 3$.

Example 4.1 The group algebra $F(Q_{12} \times C_3)$ has 12 linear and 6 non linear irreducible idempotents.

Proof. Generalized quaternion group $Q_{12} = \langle a, b : a^6 = 1, a^3 = b^2, ab = ba^{-1} \rangle$ and cyclic group C_3 =$\{1, g, g^2\}$. Conjugacy classes of Q_{12} are {1}, $\{a^3\}$,{a, a^5}, {a^2, a^4}, {b, a^2b, a^4b}, {ab,a^3b, a^5b} so Q_{12} consisting of six irreducible characters and C_3 has 3 irreducible characters therefore the group algebra $F(Q_{12} \times C_3)$ has 18 irreducible idempotents.

Notations:

$\overline{C_1} = \sum_{r=0,2,4}(a^r, 1)$; $\overline{C_2} = \sum_{r=1,3,5}(a^r, 1)$; $\overline{C_3} = \sum_{r=0,2,4}(a^r, g)$

$\overline{C_4} = \sum_{r=1,3,5}(a^r, g)$; $\overline{C_5} = \sum_{r=0,2,4}(a^r, g^2)$; $\overline{C_6} = \sum_{r=1,3,5}(a^r, g^2)$

$\overline{C_7} = \sum_{r=0,2,4}(a^r b, 1)$; $\overline{C_8} = \sum_{r=1,3,5}(a^r b, 1)$; $\overline{C_9} = \sum_{r=0,2,4}(a^r b, g)$

$\overline{C_{10}} = \sum_{r=1,3,5}(a^r b, g)$; $\overline{C_{11}} = \sum_{r=0,2,4}(a^r b, g^2)$; $\overline{C_{12}} = \sum_{r=1,3,5}(a^r b, g^2)$

The linear idempotents of $F(Q_{12} \times C_3)$ are given by

$e_1 = \frac{1}{36}\sum_{l=1}^{12} \overline{C}_l$

$e_2 = \frac{1}{36}(\sum_{l=1}^{6} \overline{C}_l - \sum_{l=7}^{12} \overline{C}_l)$

$e_3 = \frac{1}{36}(\sum_{l=1}^{6} \overline{C}_l\ (-1)^{l+1} + i\sum_{l=7}^{12} \overline{C}_l\ (-1)^{l})$

$e_4 = \frac{1}{36}(\sum_{l=1}^{6} \overline{C}_l\ (-1)^{l+1} + i\sum_{l=7}^{12} \overline{C}_l\ (-1)^{l+1})$

$e_5 = \frac{1}{36}[\overline{C_1} + \overline{C_2} + \omega^2(\overline{C_3} + \overline{C_4}) + \omega(\overline{C_5} + \overline{C_6}) + \overline{C_7} + \overline{C_8} + \omega^2(\overline{C_9} + \overline{C_{10}}) + \omega(\overline{C_{11}} + \overline{C_{12}})]$

$e_6 = \frac{1}{36}[\overline{C_1} + \overline{C_2} + \omega^2(\overline{C_3} + \overline{C_4}) + \omega(\overline{C_5} + \overline{C_6}) - \overline{C_7} - \overline{C_8} - \omega^2(\overline{C_9} + \overline{C_{10}}) - \omega(\overline{C_{11}} + \overline{C_{12}})]$

$e_7 = \frac{1}{36}[\overline{C_1} - \overline{C_2} + \omega^2(\overline{C_3} - \overline{C_4}) + \omega(\overline{C_5} - \overline{C_6}) + i(-\overline{C_7} + \overline{C_8}) + i\omega^2(-\overline{C_9} + \overline{C_{10}}) + i\omega(-\overline{C_{11}} + \overline{C_{12}})]$

$e_8 = \frac{1}{36}[\overline{C_1} - \overline{C_2} + \omega^2(\overline{C_3} - \overline{C_4}) + \omega(\overline{C_5} - \overline{C_6}) + i(\overline{C_7} - \overline{C_8}) + i\omega^2(\overline{C_9} - \overline{C_{10}}) + i\omega(\overline{C_{11}} - \overline{C_{12}})]$

$e_9 = \frac{1}{36}[\overline{C_1} + \overline{C_2} + \omega(\overline{C_3} + \overline{C_4}) + \omega^2(\overline{C_5} + \overline{C_6}) + \overline{C_7} + \overline{C_8} + \omega(\overline{C_9} + \overline{C_{10}}) + \omega^2(\overline{C_{11}} + \overline{C_{12}})]$

$e_{10} = \frac{1}{36}[\overline{C_1} + \overline{C_2} + \omega(\overline{C_3} + \overline{C_4}) + \omega^2(\overline{C_5} + \overline{C_6}) - \overline{C_7} - \overline{C_8} - \omega(\overline{C_9} + \overline{C_{10}}) - \omega^2(\overline{C_{11}} + \overline{C_{12}})]$

$e_{11} = \frac{1}{36}[\overline{C_1} - \overline{C_2} + \omega(\overline{C_3} - \overline{C_4}) + \omega^2(\overline{C_5} - \overline{C_6}) + i(-\overline{C_7} + \overline{C_8}) + i\omega(-\overline{C_9} + \overline{C_{10}}) + i\omega^2(-\overline{C_{11}} + \overline{C_{12}})]$

$e_{12} = \frac{1}{36}[\overline{C_1} - \overline{C_2} + \omega(\overline{C_3} - \overline{C_4}) + \omega^2(\overline{C_5} - \overline{C_6}) + i(\overline{C_7} - \overline{C_8}) + i\omega(\overline{C_9} - \overline{C_{10}}) + i\omega^2(\overline{C_{11}} - \overline{C_{12}})]$

and non linear idempotents are given by

$e_{13} = \frac{1}{18}[2\sum_{r=0}^{2}(1, g^r) + 2\sum_{r=0}^{2}(a^3, g^r) - \sum_{l=0}^{2}\sum_{r=1,2,4,5}(a^r, g^l)]$

$e_{14} = \frac{1}{18}[2\sum_{r=0}^{2}(1, g^r) - 2\sum_{r=0}^{2}(a^3, g^r) + \sum_{\substack{l=0\\r=1,5}}^{2}(a^r, g^l) - \sum_{\substack{l=0\\r=2,4}}^{2}(a^r, g^l)]$

$e_{15} = \frac{1}{18}[2\sum_{r=0,3}(a^r, 1) + 2\omega^2\sum_{r=0,3}(a^r, g) + 2\omega\sum_{r=0,3}(a^r, g^2) - \sum_{r=1,2,4,5}(a^r, 1) - \omega^2\sum_{r=1,2,4,5}(a^r, g) - \omega\sum_{r=1,2,4,5}(a^r, g^2)]$

$e_{16} = \frac{1}{18}[2(1,1) - 2(a^3, 1) + 2\omega^2\{(1, g) - (a^3, g)\} + 2\omega\{(1, g^2) - (a^3, g^2)\} + \sum_{r=1,5}(a^r, 1) + \omega^2\sum_{r=1,5}(a^r, g) + \omega\sum_{r=1,5}(a^r, g^2) - \sum_{r=2,4}(a^r, 1) - \omega^2\sum_{r=2,4}(a^r, g) - \omega\sum_{r=2,4}(a^r, g^2)]$

$e_{17} = \frac{1}{18}[2\sum_{r=0,3}(a^r, 1) + 2\omega\sum_{r=0,3}(a^r, g) + 2\omega^2\sum_{r=0,3}(a^r, g^2) - \sum_{r=1,2,4,5}(a^r, 1) - \omega\sum_{r=1,2,4,5}(a^r, g) - \omega^2\sum_{r=1,2,4,5}(a^r, g^2)]$

$e_{18} = \frac{1}{18}[2(1,1) - 2(a^3, 1) + 2\omega\{(1, g) - (a^3, g)\} + 2\omega^2\{(1, g^2) - (a^3, g^2)\} + \sum_{r=1,5}(a^r, 1) + \omega\sum_{r=1,5}(a^r, g) + \omega^2\sum_{r=1,5}(a^r, g^2) - \sum_{r=2,4}(a^r, 1) - \omega\sum_{r=2,4}(a^r, g) - \omega^2\sum_{r=2,4}(a^r, g^2)]$

where ω is cube root of unity and i is 4^{th} root of unity.

Remark: In this case the set of all irreducible idempotents $E = \{e_1, e_2, e_3, \dots, e_{18}\}$ and group code $I_\mu = \{u \in F(Q_{12} \times C_3): ue = 0 \ \forall e \in \mu\}$ is generated by $\{E \setminus \mu\}$. e.g. The group code $I_{\{e_1,e_2,e_3,e_4\}}$ is generated by the idempotents $\{e_5, e_6, \dots, e_{18}\}$.

4.2 The minimum distance and dimension of the group codes are

(i) $d(I_{\{e_i\}})$= 2 and $dim(I_{\{e_i\}})$ =35 for $1 \le i \le 12$
(ii) $d(I_\beta)$ = 2 and $dim(I_\beta)$ = $36 - |\beta|$ where $\beta \subseteq \{e_1, \dots, e_{12}\}$, $|\beta| \ge 2$
(iii) $d(I_{\{e_i\}})$= 3 and $dim(I_{\{e_i\}})$= 32 for $13 \le i \le 18$
(iv) $d(I_{\{e_l,e_m\}}) = 6$ and $dim(I_{\{e_l,e_m\}})$ = 28 for $13 \le l, m \le 18$
(v) $d(I_\beta)$ = 6 and $dim(I_\beta)$ = $36 - 2^2|\beta|$ where $\beta \subseteq \{e_{13}, \dots, e_{18}\}$ and $|\beta| \ge 3$
(vi) $d(I_{\{e_l,e_m\}}) = 2$ and $dim(I_{\{e_l,e_m\}}) = 31$ for l=1,2,5,6,9,10 and m= 13,15,17
(vii) $d(I_{\{e_l,e_m\}}) = 2$ and $dim(I_{\{e_l,e_m\}}) = 31$ for l =3,4,7,8,11,12 and m= 14,16,18
(viii) $d(I_{\{e_l,e_m\}}) = 6$ and $dim(I_{\{e_l,e_m\}}) = 31$ for l=1,2,5,6,9,10 and m= 14,16,18
(ix) $d(I_{\{e_l,e_m\}}) = 6$ and $dim(I_{\{e_l,e_m\}}) = 31$ for l =3,4,7,8,11,12 and m= 13,15,17

Proof. Let $u = \sum_{r=0}^{5}\lambda_{r+1}(a^r, 1) + \sum_{r=0}^{5}\lambda_{7+r}(a^r, g) + \sum_{r=0}^{5}\lambda_{13+r}(a^r, g^2) + \sum_{r=0}^{5}\lambda_{19+r}(a^r b, 1) + \sum_{r=0}^{5}\lambda_{25+r}(a^r b, g) + \sum_{r=0}^{5}\lambda_{31+r}(a^r b, g^2)$ be any element of $F(Q_{12} \times C_3)$ then

$ue_1 = (\sum_{i=1}^{36}\lambda_i)e_1$ (4.1)

$ue_2 = (\sum_{i=1}^{18}\lambda_i - \sum_{i=19}^{36}\lambda_i)e_2$ (4.2)

$ue_3 = (\sum_{r=1}^{18}\lambda_r(-1)^{r+1} + i\sum_{r=19}^{36}\lambda_r(-1)^{r+1})e_3$ (4.3)

$ue_4 = (\sum_{r=1}^{18}\lambda_r(-1)^{r+1} + i\sum_{r=19}^{36}\lambda_r(-1)^{r})e_4$ (4.4)

ue_5 = $(\sum_{i=1}^{6}\lambda_i + \omega\sum_{i=7}^{12}\lambda_i + \omega^2\sum_{i=13}^{18}\lambda_i + \sum_{i=19}^{24}\lambda_i + \omega\sum_{i=25}^{30}\lambda_i + \omega^2\sum_{i=31}^{36}\lambda_i)e_5$ (4.5)

ue_6 = $(\sum_{i=1}^{6}\lambda_i + \omega\sum_{i=7}^{12}\lambda_i + \omega^2\sum_{i=13}^{18}\lambda_i - \sum_{i=19}^{24}\lambda_i - \omega\sum_{i=25}^{30}\lambda_i - \omega^2\sum_{i=31}^{36}\lambda_i)e_6$ (4.6)

ue_7 = $(\sum_{r=1}^{6}(-1)^{r+1}\lambda_r + \omega\sum_{r=7}^{12}(-1)^{r+1}\lambda_r + \omega^2\sum_{r=13}^{18}(-1)^{r+1}\lambda_r + i\sum_{r=19}^{24}(-1)^{r+1}\lambda_r + i\omega\sum_{r=25}^{30}(-1)^{r+1}\lambda_r + i\omega^2\sum_{r=31}^{36}(-1)^{r+1}\lambda_r)e_7$ (4.7)

ue_8 = $(\sum_{r=1}^{6}(-1)^{r+1}\lambda_r + \omega\sum_{r=7}^{12}(-1)^{r+1}\lambda_r + \omega^2\sum_{r=13}^{18}(-1)^{r+1}\lambda_r + i\sum_{r=19}^{24}(-1)^{r}\lambda_r + i\omega\sum_{r=25}^{30}(-1)^{r}\lambda_r + i\omega^2\sum_{r=31}^{36}(-1)^{r}\lambda_r)e_7$ (4.8)

ue_9 = $(\sum_{i=1}^{6}\lambda_i + \omega^2\sum_{i=7}^{12}\lambda_i + \omega\sum_{i=13}^{18}\lambda_i + \sum_{i=19}^{24}\lambda_i + \omega^2\sum_{i=25}^{30}\lambda_i + \omega\sum_{i=31}^{36}\lambda_i)e_9$ (4.9)

ue_{10} = $(\sum_{i=1}^{6}\lambda_i + \omega^2\sum_{i=7}^{12}\lambda_i + \omega\sum_{i=13}^{18}\lambda_i - \sum_{i=19}^{24}\lambda_i - \omega^2\sum_{i=25}^{30}\lambda_i - \omega\sum_{i=31}^{36}\lambda_i)e_{10}$ (4.10)

ue_{11} = $(\sum_{r=1}^{6}(-1)^{r+1}\lambda_r + \omega^2\sum_{r=7}^{12}(-1)^{r+1}\lambda_r + \omega\sum_{r=13}^{18}(-1)^{r+1}\lambda_r + i\sum_{r=19}^{24}(-1)^{r+1}\lambda_r + i\omega^2\sum_{r=25}^{30}(-1)^{r+1}\lambda_r + i\omega\sum_{r=31}^{36}(-1)^{r+1}\lambda_r)e_{11}$ (4.11)

ue_{12} = $(\sum_{r=1}^{6}(-1)^{r+1}\lambda_r + \omega^2\sum_{r=7}^{12}(-1)^{r+1}\lambda_r + \omega\sum_{r=13}^{18}(-1)^{r+1}\lambda_r + i\sum_{r=19}^{24}(-1)^{r}\lambda_r + i\omega^2\sum_{r=25}^{30}(-1)^{r}\lambda_r + i\omega\sum_{r=31}^{36}(-1)^{r}\lambda_r)e_{12}$ (4.12)

$ue_{13} = \frac{1}{18}[(2\lambda_1 - \lambda_2 - \lambda_3 + 2\lambda_4 - \lambda_5 - \lambda_6 + 2\lambda_7 - \lambda_8 - \lambda_9 + 2\lambda_{10} - \lambda_{11} - \lambda_{12} + 2\lambda_{13} - \lambda_{14} - \lambda_{15} + 2\lambda_{16} - \lambda_{17} - \lambda_{18})\{(1,1) + (1,g) + (1,g^2) + (a^3,1) + (a^3,g) + (a^3,g^2)\} + (-\lambda_1 + 2\lambda_2 - \lambda_3 - \lambda_4 + 2\lambda_5 - \lambda_6 - \lambda_7 + 2\lambda_8 - \lambda_9 - \lambda_{10} + 2\lambda_{11} - \lambda_{12} - \lambda_{13} + 2\lambda_{14} - \lambda_{15} - \lambda_{16} + 2\lambda_{17} - \lambda_{18})\{(a,1) + (a,g) + (a,g^2) + (a^4,1) + (a^4,g) + (a^4,g^2)\} + (-\lambda_1 - \lambda_2 + 2\lambda_3 - \lambda_4 - \lambda_5 + 2\lambda_6 - \lambda_7 - \lambda_8 + 2\lambda_9 - \lambda_{10} - \lambda_{11} + 2\lambda_{12} - \lambda_{13} - \lambda_{14} + 2\lambda_{15} - \lambda_{16} - \lambda_{17} + 2\lambda_{18})\{(a^2,1) + (a^2,g) + (a^2,g^2) + (a^5,1) + (a^5,g) + (a^5,g^2)\} + (2\lambda_{19} - \lambda_{20} - \lambda_{21} + 2\lambda_{22} - \lambda_{23} - \lambda_{24} + 2\lambda_{25} - \lambda_{26} - \lambda_{27} + 2\lambda_{28} - \lambda_{29} - \lambda_{30} + 2\lambda_{31} - \lambda_{32} - \lambda_{33} + 2\lambda_{34} - \lambda_{35} - \lambda_{36})\{(b,1) + (b,g) + (b,g^2) + (a^3b,1) + (a^3b,g) + (a^3b,g^2)\} + (-\lambda_{19} + 2\lambda_{20} - \lambda_{21} - \lambda_{22} + 2\lambda_{23} - \lambda_{24} - \lambda_{25} + 2\lambda_{26} - \lambda_{27} - \lambda_{28} + 2\lambda_{29} - \lambda_{30} - \lambda_{31} + 2\lambda_{32} - \lambda_{33} - \lambda_{34} + 2\lambda_{35} - \lambda_{36})\{(ab,1) + (ab,g) + (ab,g^2) + (a^4b,1) + (a^4b,g) + (a^4b,g^2)\} + (-\lambda_{19} - \lambda_{20} + 2\lambda_{21} - \lambda_{22} - \lambda_{23} + 2\lambda_{24} - \lambda_{25} - \lambda_{26} + 2\lambda_{27} - \lambda_{28} - \lambda_{29} + 2\lambda_{30} - \lambda_{31} - \lambda_{32} + 2\lambda_{33} - \lambda_{34} - \lambda_{35} + 2\lambda_{36})\{(a^2b,1) + (a^2b,g) + (a^2b,g^2) + (a^5b,1) + (a^5b,g) + (a^5b,g^2)\}]$ (4.13)

$ue_{14} = \frac{1}{18}[(2\lambda_1 + \lambda_2 - \lambda_3 - 2\lambda_4 - \lambda_5 + \lambda_6 + 2\lambda_7 + \lambda_8 - \lambda_9 - 2\lambda_{10} - \lambda_{11} + \lambda_{12} + 2\lambda_{13} + \lambda_{14} - \lambda_{15} - 2\lambda_{16} - \lambda_{17} + \lambda_{18})\{(1,1) + (1,g) + (1,g^2) - (a^3,1) - (a^3,g) - (a^3,g^2)\} + (\lambda_1 + 2\lambda_2 + \lambda_3 - \lambda_4 - 2\lambda_5 - \lambda_6 + \lambda_7 + 2\lambda_8 + \lambda_9 - \lambda_{10} - 2\lambda_{11} - \lambda_{12} + \lambda_{13} + 2\lambda_{14} + \lambda_{15} - \lambda_{16} - 2\lambda_{17} - \lambda_{18})\{(a,1) + (a,g) + (a,g^2) - (a^4,1) - (a^4,g) - (a^4,g^2)\} + (-\lambda_1 + \lambda_2 + 2\lambda_3 + \lambda_4 - \lambda_5 - 2\lambda_6 - \lambda_7 + \lambda_8 + 2\lambda_9 + \lambda_{10} - \lambda_{11} - 2\lambda_{12} - \lambda_{13} + \lambda_{14} + 2\lambda_{15} + \lambda_{16} - \lambda_{17} - 2\lambda_{18})\{(a^2,1) + (a^2,g) + (a^2,g^2) - (a^5,1) - (a^5,g) - (a^5,g^2)\}$ (4.14)

$ue_{15} = \frac{1}{18}[\{2\lambda_1 - \lambda_2 - \lambda_3 + 2\lambda_4 - \lambda_5 - \lambda_6 + \omega(2\lambda_7 - \lambda_8 - \lambda_9 + 2\lambda_{10} - \lambda_{11} - \lambda_{12}) + \omega^2(2\lambda_{13} - \lambda_{14} - \lambda_{15} + 2\lambda_{16} - \lambda_{17} - \lambda_{18})\}\{(1,1) + (a^3,1)\} + \{\omega^2(2\lambda_1 - \lambda_2 - \lambda_3 + 2\lambda_4 - \lambda_5 - \lambda_6) + (2\lambda_7 - \lambda_8 - \lambda_9 + 2\lambda_{10} - \lambda_{11} - \lambda_{12}) + \omega(2\lambda_{13} - \lambda_{14} - \lambda_{15} + 2\lambda_{16} - \lambda_{17} - \lambda_{18})\}\{(1,g) + (a^3,g)\} + \{\omega(2\lambda_1 - \lambda_2 - \lambda_3 + 2\lambda_4 - \lambda_5 - \lambda_6) + \omega^2(2\lambda_7 - \lambda_8 - \lambda_9 + 2\lambda_{10} - \lambda_{11} - \lambda_{12}) + (2\lambda_{13} - \lambda_{14} - \lambda_{15} + 2\lambda_{16} - \lambda_{17} - \lambda_{18})\}\{(1,g^2) + (a^3,g^2)\}$ +

$\{-\lambda_1 + 2\lambda_2 - \lambda_3 - \lambda_4 + 2\lambda_5 - \lambda_6 + \omega(-\lambda_7 + 2\lambda_8 - \lambda_9 - \lambda_{10} + 2\lambda_{11} - \lambda_{12}) + \omega^2(-\lambda_{13} + 2\lambda_{14} - \lambda_{15} - \lambda_{16} + 2\lambda_{17} - \lambda_{18})\}\{(a,1) + (a^4,1)\}$ +

$\{\omega^2(-\lambda_1 + 2\lambda_2 - \lambda_3 - \lambda_4 + 2\lambda_5 - \lambda_6) + (-\lambda_7 + 2\lambda_8 - \lambda_9 - \lambda_{10} + 2\lambda_{11} - \lambda_{12}) + \omega(-\lambda_{13} + 2\lambda_{14} - \lambda_{15} - \lambda_{16} + 2\lambda_{17} - \lambda_{18})\}\{(a,g) + (a^4,g)\}$ +

$\{\omega(-\lambda_1 + 2\lambda_2 - \lambda_3 - \lambda_4 + 2\lambda_5 - \lambda_6) + \omega^2(-\lambda_7 + 2\lambda_8 - \lambda_9 - \lambda_{10} + 2\lambda_{11} - \lambda_{12}) + (-\lambda_{13} + 2\lambda_{14} - \lambda_{15} - \lambda_{16} + 2\lambda_{17} - \lambda_{18})\}\{(a,g^2) + (a^4,g^2)\}$ +

$\{-\lambda_1 - \lambda_2 + 2\lambda_3 - \lambda_4 - \lambda_5 + 2\lambda_6 + \omega(-\lambda_7 - \lambda_8 + 2\lambda_9 - \lambda_{10} - \lambda_{11} + 2\lambda_{12}) + \omega^2(-\lambda_{13} - \lambda_{14} + 2\lambda_{15} - \lambda_{16} - \lambda_{17} + 2\lambda_{18})\}\{(a^2,1) + (a^5,1)\}$ +

$\{\omega^2(-\lambda_1 - \lambda_2 + 2\lambda_3 - \lambda_4 - \lambda_5 + 2\lambda_6) + (-\lambda_7 - \lambda_8 + 2\lambda_9 - \lambda_{10} - \lambda_{11} + 2\lambda_{12}) + \omega(-\lambda_{13} - \lambda_{14} + 2\lambda_{15} - \lambda_{16} - \lambda_{17} + 2\lambda_{18})\}\{(a^2,g) + (a^5,g)\}$ + $\{\omega(-\lambda_1 - \lambda_2 + 2\lambda_3 - \lambda_4 - \lambda_5 + 2\lambda_6) + \omega^2(-\lambda_7 - \lambda_8 +$

$2\lambda_9-\lambda_{10}-\lambda_{11}+2\lambda_{12})+(-\lambda_{13}-\lambda_{14}+2\lambda_{15}-\lambda_{16}-\lambda_{17}+2\lambda_{18})\}\{(a^2,g^2)+(a^5,g^2)\}$ + $\{2\lambda_{19}-\lambda_{20}-\lambda_{21}+2\lambda_{22}-\lambda_{23}-\lambda_{24}+\omega(2\lambda_{25}-\lambda_{26}-\lambda_{27}+2\lambda_{28}-\lambda_{29}-\lambda_{30})+\omega^2(2\lambda_{31}-\lambda_{32}-\lambda_{33}+2\lambda_{34}-\lambda_{35}-\lambda_{36})\}\{(b,1)+(a^3b,1)\}$ + $\{\omega^2(2\lambda_{19}-\lambda_{20}-\lambda_{21}+2\lambda_{22}-\lambda_{23}-\lambda_{24})+2\lambda_{25}-\lambda_{26}-\lambda_{27}+2\lambda_{28}-\lambda_{29}-\lambda_{30}+\omega(2\lambda_{31}-\lambda_{32}-\lambda_{33}+2\lambda_{34}-\lambda_{35}-\lambda_{36})\}\{(b,g)+(a^3b,g)\}$ +

$\{\omega(2\lambda_{19}-\lambda_{20}-\lambda_{21}+2\lambda_{22}-\lambda_{23}-\lambda_{24})+\omega^2(2\lambda_{25}-\lambda_{26}-\lambda_{27}+2\lambda_{28}-\lambda_{29}-\lambda_{30})+2\lambda_{31}-\lambda_{32}-\lambda_{33}+2\lambda_{34}-\lambda_{35}-\lambda_{36})\}\{(b,g^2)+(a^3b,g^2)\}$ +

$\{-\lambda_{19}+2\lambda_{20}-\lambda_{21}-\lambda_{22}+2\lambda_{23}-\lambda_{24}+\omega(-\lambda_{25}+2\lambda_{26}-\lambda_{27}-\lambda_{28}+2\lambda_{29}-\lambda_{30})+\omega^2(-\lambda_{31}+2\lambda_{32}-\lambda_{33}-\lambda_{34}+2\lambda_{35}-\lambda_{36})\}\{(ab,1)+(a^4b,1)\}$ +

$\{\omega^2(-\lambda_{19}+2\lambda_{20}-\lambda_{21}-\lambda_{22}+2\lambda_{23}-\lambda_{24})-\lambda_{25}+2\lambda_{26}-\lambda_{27}-\lambda_{28}+2\lambda_{29}-\lambda_{30}+\omega(-\lambda_{31}+2\lambda_{32}-\lambda_{33}-\lambda_{34}+2\lambda_{35}-\lambda_{36})\}\{(ab,g)+(a^4b,g)\}$ +

$\{\omega(-\lambda_{19}+2\lambda_{20}-\lambda_{21}-\lambda_{22}+2\lambda_{23}-\lambda_{24})+\omega^2(-\lambda_{25}+2\lambda_{26}-\lambda_{27}-\lambda_{28}+2\lambda_{29}-\lambda_{30})-\lambda_{31}+2\lambda_{32}-\lambda_{33}-\lambda_{34}+2\lambda_{35}-\lambda_{36})\{(ab,g^2)+(a^4b,g^2)\}$ +

$\{-\lambda_{19}-\lambda_{20}+2\lambda_{21}-\lambda_{22}-\lambda_{23}+2\lambda_{24}+\omega(-\lambda_{25}-\lambda_{26}+2\lambda_{27}-\lambda_{28}-\lambda_{29}+2\lambda_{30})++\omega^2(-\lambda_{31}-\lambda_{32}+2\lambda_{33}-\lambda_{34}-\lambda_{35}+2\lambda_{36})\}\ \{(a^2b,1)+(a^5b,1)\}$ +

$\{\omega^2(-\lambda_{19}-\lambda_{20}+2\lambda_{21}-\lambda_{22}-\lambda_{23}+2\lambda_{24})-\lambda_{25}-\lambda_{26}+2\lambda_{27}-\lambda_{28}-\lambda_{29}+2\lambda_{30}+\omega(-\lambda_{31}-\lambda_{32}+2\lambda_{33}-\lambda_{34}-\lambda_{35}+2\lambda_{36})\}\{(a^2b,g)+(a^5b,g)\}$ + $\{\omega(-\lambda_{19}-\lambda_{20}+2\lambda_{21}-\lambda_{22}-\lambda_{23}+2\lambda_{24})+\omega^2(-\lambda_{25}-\lambda_{26}+2\lambda_{27}-\lambda_{28}-\lambda_{29}+2\lambda_{30})-\lambda_{31}-\lambda_{32}+2\lambda_{33}-\lambda_{34}-\lambda_{35}+2\lambda_{36}\}\{(a^2b,g^2)+(a^5b,g^2)\}]$ (4.15) Similarly we calculate ue_i for i=16,17,18.

(i) Using the equations (4.1) to (4.12) the code word of weight two $\lambda_1(1,1)+\lambda_3(a^2,1)\in I_{\{e_i\}}$ if and only if $\lambda_1=-\lambda_3$ therefore $d(I_{\{e_i\}})$ =2 and $dim(I_{\{e_i\}})$ =36−1=35. So $I_{\{e_i\}}$ is (36,35,2) group code.

(iv) Using the equations (4.13) to (4.15) the code word of weight six, $\lambda_1(1,1)+\lambda_2(a,1)+\lambda_3(a^2,1)+\lambda_4(a^3,1)+\lambda_5(a^4,1)+\lambda_6(a^5,1)\in I_{\{e_l,e_m\}}$ if and only if $\lambda_1=\lambda_2=\lambda_3=\lambda_4=\lambda_5=\lambda_6$. Therefore $d(I_{\{e_l,e_m\}})=6$ and using remark 1.1, $dim(I_{\{e_l,e_m\}})=36-2.2^2=28$. The other parts can be proved in a similar manner.

5 Codes over finite group algebra

In the previous sections, we constructed the codes over the group algebra FG where F is field of characteristic zero. Now we consider the group algebra F_qG where $F=F_q$ denotes the finite field with q elements and $q=p^s$ where p is odd prime and $gcd(char(F),|G|)=1$. To illustrate the concept of finite group algebra we give some examples.

Example 5.1 Idempotents of the group algebra $F_q(Q_{12}\times C_3)$

If we consider the group algebra $F_5(Q_{12}\times C_3)$ (see 4.1) then −1 is quadratic residue mod5, $2^2\equiv-1(mod5)$. Further ω,ω² $\notin F_5$ so there exist an extension $F_{5^2}(=GF(5^2))=\frac{F_5[x]}{<x^2+x+1>}$ of the field F_5 where x^2+x+1 is minimal polynomial of ω and irreducible over F_5.

An arbitrary element of $GF(5^2)$ is of the form $ax+b+<x^2+x+1>\ \ a,b\in F_5$.

Let the element $x+<x^2+x+1>$ be denoted by α. Then $\beta=\alpha+2$ is primitive element of $GF(5^2)$ and $\beta^{16}=\omega$ and $\beta^8=\omega^2$.

The idempotents of $F_{5^2}(Q_{12}\times C_3)$ are given by

$e_1=\sum_{l=1}^{12}\overline{C}_l$

$e_2=(\sum_{l=1}^{6}\overline{C}_l-\sum_{l=7}^{12}\overline{C}_l)$

$e_3=(\sum_{l=1}^{6}\overline{C}_l\,(-1)^{l+1}+2\sum_{l=7}^{12}\overline{C}_l\,(-1)^{l})$

$e_4=(\sum_{l=1}^{6}\overline{C}_l\,(-1)^{l+1}+2\sum_{l=7}^{12}\overline{C}_l\,(-1)^{l+1})$

$e_5 = [\overline{C_1} + \overline{C_2} + \omega^2(\overline{C_3} + \overline{C_4}) + \omega(\overline{C_5} + \overline{C_6}) + \overline{C_7} + \overline{C_8} + \omega^2(\overline{C_9} + \overline{C_{10}}) + \omega(\overline{C_{11}} + \overline{C_{12}})]$

$e_6 = [\overline{C_1} + \overline{C_2} + \omega^2(\overline{C_3} + \overline{C_4}) + \omega(\overline{C_5} + \overline{C_6}) - \overline{C_7} - \overline{C_8} - \omega^2(\overline{C_9} + \overline{C_{10}}) - \omega(\overline{C_{11}} + \overline{C_{12}})]$

$e_7 = [\overline{C_1} - \overline{C_2} + \omega^2(\overline{C_3} - \overline{C_4}) + \omega(\overline{C_5} - \overline{C_6}) + 2(-\overline{C_7} + \overline{C_8}) + 2\omega^2(-\overline{C_9} + \overline{C_{10}}) + 2\omega(-\overline{C_{11}} + \overline{C_{12}})]$

$e_8 = [\overline{C_1} - \overline{C_2} + \omega^2(\overline{C_3} - \overline{C_4}) + \omega(\overline{C_5} - \overline{C_6}) + 2(\overline{C_7} - \overline{C_8}) + 2\omega^2(\overline{C_9} - \overline{C_{10}}) + 2\omega(\overline{C_{11}} - \overline{C_{12}})]$

$e_9 = [\overline{C_1} + \overline{C_2} + \omega(\overline{C_3} + \overline{C_4}) + \omega^2(\overline{C_5} + \overline{C_6}) + \overline{C_7} + \overline{C_8} + \omega(\overline{C_9} + \overline{C_{10}}) + \omega^2(\overline{C_{11}} + \overline{C_{12}})]$

$e_{10} = [\overline{C_1} + \overline{C_2} + \omega(\overline{C_3} + \overline{C_4}) + \omega^2(\overline{C_5} + \overline{C_6}) - \overline{C_7} - \overline{C_8} - \omega(\overline{C_9} + \overline{C_{10}}) - \omega^2(\overline{C_{11}} + \overline{C_{12}})]$

$e_{11} = [\overline{C_1} - \overline{C_2} + \omega(\overline{C_3} - \overline{C_4}) + \omega^2(\overline{C_5} - \overline{C_6}) + 2(-\overline{C_7} + \overline{C_8}) + 2\omega(-\overline{C_9} + \overline{C_{10}}) + 2\omega^2(-\overline{C_{11}} + \overline{C_{12}})]$

$e_{12} = [\overline{C_1} - \overline{C_2} + \omega(\overline{C_3} - \overline{C_4}) + \omega^2(\overline{C_5} - \overline{C_6}) + 2(\overline{C_7} - \overline{C_8}) + 2\omega(\overline{C_9} - \overline{C_{10}}) + 2\omega^2(\overline{C_{11}} - \overline{C_{12}})]$

$e_{13} = \frac{1}{18}[2\sum_{r=0}^{2}(1, g^r) + 2\sum_{r=0}^{2}(a^3, g^r) - \sum_{l=0}^{2}\sum_{r=1,2,4,5}(a^r, g^l)]$

$e_{14} = \frac{1}{18}[2\sum_{r=0}^{2}(1, g^r) - 2\sum_{r=0}^{2}(a^3, g^r) + \sum_{\substack{l=0\\r=1,5}}^{2}(a^r, g^l) - \sum_{\substack{l=0\\r=2,4}}^{2}(a^r, g^l)]$

$e_{15} = \frac{1}{18}[2\sum_{r=0,3}(a^r, 1) + 2\omega^2\sum_{r=0,3}(a^r, g) + 2\omega\sum_{r=0,3}(a^r, g^2) - \sum_{r=1,2,4,5}(a^r, 1) - \omega^2\sum_{r=1,2,4,5}(a^r, g) - \omega\sum_{r=1,2,4,5}(a^r, g^2)]$

$e_{16} = \frac{1}{18}[2(1,1) - 2(a^3, 1) + 2\omega^2\{(1, g) - (a^3, g)\} + 2\omega\{(1, g^2) - (a^3, g^2)\} + \sum_{r=1,5}(a^r, 1) + \omega^2\sum_{r=1,5}(a^r, g) + \omega\sum_{r=1,5}(a^r, g^2) - \sum_{r=2,4}(a^r, 1) - \omega^2\sum_{r=2,4}(a^r, g) - \omega\sum_{r=2,4}(a^r, g^2)]$

$e_{17} = \frac{1}{18}[2\sum_{r=0,3}(a^r, 1) + 2\omega\sum_{r=0,3}(a^r, g) + 2\omega^2\sum_{r=0,3}(a^r, g^2) - \sum_{r=1,2,4,5}(a^r, 1) - \omega\sum_{r=1,2,4,5}(a^r, g) - \omega^2\sum_{r=1,2,4,5}(a^r, g^2)]$

$e_{18} = \frac{1}{18}[2(1,1) - 2(a^3, 1) + 2\omega\{(1, g) - (a^3, g)\} + 2\omega^2\{(1, g^2) - (a^3, g^2)\} + \sum_{r=1,5}(a^r, 1) + \omega\sum_{r=1,5}(a^r, g) + \omega^2\sum_{r=1,5}(a^r, g^2) - \sum_{r=2,4}(a^r, 1) - \omega\sum_{r=2,4}(a^r, g) - \omega^2\sum_{r=2,4}(a^r, g^2)]$.

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