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QUANTUM CORRELATION OF A TWO-COMPONENT BOSE-EINSTEIN CONDENSATE IN AN OPTICAL CAVITY

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Abstract: We investigate the dynamics of a two-component Bose-Einstein condensate dispersively coupled by an optical cavity. Following the approach of D. Nagy et. al. (Eur. Phys. J. D 55, 659, 2009), we derive the corresponding master equation for the quadratures of a two-component condensate by adiabatically eliminating the cavity field. The existence of resulting diffusion and friction makes the dynamics differ significantly from the non-dissipative case in the classically bistable regime. We give numerical results such as linear entropy and correlations between the two quadratures to demonstrate our conclusions.

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Recent success in trapping condensed bosonic atoms inside an optical cavity has attracted much interest in investigations of cavity quantum electrodynamics and ultracold atomic physics [1 – 16]. Compared to the scenario where a Bose-Einstein condensate is placed inside an optical lattice in free space, the quantized nature of the cavity photon field plays a strikingly different role. The cavity field is dispersively coupled to the condensate and affects the motion of the condensate which in turn acts back to the cavity field and modifies the photon distribution inside the cavity. The coupling between the condensate and the cavity field in many aspects resembles the canonical optomechanics. For example, bistable behaviors have been experimentally observed which demonstrates the intrinsic nonlinearity of the system [7, 15].

In this paper, we investigate the dynamics of a twocomponent condensate in an optical cavity. We extend the previous work for a single-component condensate to a two-component condensate [12]. By taking the leakage of cavity photons into consideration, we derive the corresponding master equation along similar lines to Ref. [12]. Because both components are simultaneously coupled to the cavity field through the radiation pressure from intra-cavity photons, we are particularly interested in the correlation between the two components.

We consider a cigar-shaped two-component Bose-Einstein condensate in a single mode high-Q optical cavity. The two components are hereafter labelled as 1 and 2. The cavity with mode frequency ω_{cav} is driven by an external laser field with frequency ω. The cavity frequency is detuned far below the atomic transition frequency $\omega_{at,\,j}$ for component j, so the excited state can be adiabatically eliminated. This results in an optical potential $U_j = g_j^2/(\omega - \omega_{at,\,j})$ for atoms of component j. g_j is the single photon Rabi frequency of the cavity. Because the condensate is tightly trapped in the transverse direction, transverse degrees of freedom are frozen. Therefore we can treat the condensate as a one dimensional system along the x-axis. We further assume a monotonic cavity mode $k = \omega_{cav}/c$ with a corresponding mode function cos (kx). The above assumptions greatly simplify our derivations and allow us to grasp essential features of the dynamics.

In the rotating frame of the pump laser, the many-body Hamiltonian for the combined condensate-cavity system is $(\hbar = 1)$

$$H = -\Delta_C a^\dagger a + i\eta\left(a^\dagger - a\right)$$
$$+ \sum_{j=1,2}\int dx \Psi_j^\dagger(x)\left[-\frac{\nabla^2}{2m_j} + U_j a^\dagger a \cos^2(kx)\right]\psi_j(x)$$
$$+ \Omega_0 \int dx\left(\Psi_1^\dagger(x)\Psi_2(x) + \Psi_1(x)\Psi_2^\dagger(x)\right) \quad \ldots (1)$$

where $\Delta_C = \omega - \omega_{cav}$. The origin is assumed to be located at the center of the cavity. a and $a^\dagger$ are annihilation and creation operators for cavity photons, respectively, satisfying $[a, a^\dagger] = 1$. η denotes the amplitude of the driving laser. $\Psi_j(x)$ and $\Psi_j^\dagger(x)$ are condensate field operators with $[\Psi_j(x), \Psi_k^\dagger(x')] = \delta_{j,k}\delta(x - x')$.$m_j$ is the atom mass for component j. For completeness, we have also included a possible conversion between the two components with a Rabi frequency Ω_0. In the above Hamiltonian, we have neglected the s-wave scattering between the atoms for the weak atom-atom interaction limit as in previous treatments, e.g. [12]. In the following, we take a similar approach as shown by Nagy et. al. [12] which also serves as a review of the method in Ref. [12].

Restricting the dynamics to the two lowest modes of the condensate, we expand the atomic field operators as $\Psi_1(x) \propto c_0 + \sqrt{2}c_2 \cos(2kx)$ and $\Psi_2(x) \propto d_0 + \sqrt{2}d_2 \cos(2kx)$ where c_0, c_2, d_0, and d_2 are the usual bosonic operators for the corresponding modes 1 and 2, respectively. The Hamiltonian can be rewritten as

$$H = \left(-\Delta_C + \frac{N_1U_1}{2} + \frac{N_2U_2}{2}\right)a^\dagger a + i\eta\left(a^\dagger - a\right)$$
$$+ 4\omega_{R,1} c_2^\dagger c_2 + \frac{\sqrt{2}U_1}{4} a^\dagger a\left(c_0^\dagger c_2 + c_0 c_2^\dagger\right)$$
$$+ 4\omega_{R,2} d_2^\dagger d_2 + \frac{\sqrt{2}U_2}{4} a^\dagger a\left(d_0^\dagger d_2 + d_0 d_2^\dagger\right)$$
$$+ \Omega_0\left(c_2^\dagger d_2 + c_2 d_2^\dagger\right) + \Omega_0\left(c_0^\dagger d_0 + c_0 d_0^\dagger\right) \quad \ldots (2)$$

where $\omega_{R,j} = k^2/(2m_j)$ is the recoil energy for component j. $c_0^\dagger c_0 + c_2^\dagger c_2 = N_1$ and $d_0^\dagger d_0 + d_2^\dagger d_2 = N_2$. N_j is the atom number of component j. We can immediately see that, for $\Omega_0 \neq 0$, both N_1 and N_2 are time dependent. This makes the dynamics rather complicated and it seems not possible for a simple analytic treatment. Therefore, we focus on the case with $\Omega_0 = 0$ in the following discussions. We also note that the above Hamiltonian is quadratic in $c_{0,2}$ and $d_{0,2}$ and in principle can be diagonalized by a unitary transformation. However, such a transformation depends on $a^\dagger a$ which actually brings more complexity to the above problem.

Therefore, we closely follow the approach which is shown in Ref. [12]. To begin with, we assume the depletion from the ground state is small, i.e. $c_0 \simeq \sqrt{N_1}$ and $d_0 \simeq \sqrt{N_2}$, and introduce the quadratures

$$X_1 = \frac{1}{\sqrt{2}}\left(c_2^\dagger + c_2\right), \qquad Y_1 = \frac{i}{\sqrt{2}}\left(c_2^\dagger + c_2\right),$$

$$X_2 = \frac{1}{\sqrt{2}}\left(d_2^\dagger + d_2\right), \qquad Y_2 = \frac{i}{\sqrt{2}}\left(d_2^\dagger - d_2\right).$$

The Hamiltonian becomes

$$H = -\tilde{\Delta}_C a^\dagger a + i\eta\left(a^\dagger - a\right)$$
$$+ \sum_{j=1,2}\left[2\omega_{R,j}\left(X_j^2 + Y_j^2\right) + u_j a^\dagger a X_j\right],$$

$$\tilde{\Delta}_C = \Delta_C - \frac{N_1U_1}{2} - \frac{N_2U_2}{2},$$

$$u_j = \frac{\sqrt{N_j}U_j}{2}. \quad \ldots (3)$$

The corresponding Heisenberg – Langevin equation is

$$\dot{a} = \left[i\left(\tilde{\Delta}_C - \sum_j u_j X_j\right) - k\right]a + \eta + \xi, \quad \ldots (4a)$$

$$\dot{X}_j = 4\omega_{R,j} Y_j, \quad \ldots (4b)$$

$$\dot{Y}_j = -4\omega_{R,j} X_j - u_j a^\dagger a \quad \ldots (4c)$$

where $\langle \xi(t)\xi^\dagger(t')\rangle = 2k\delta(t - t')$ and k is the cavity decay rate.

For typical experimental conditions with $k >> \omega_R$, we can adiabatically eliminate the cavity photon field. We then obtain

$$a(t) = \alpha(t) + \Sigma(t),$$

$$\alpha(t) = \frac{\eta}{k - i\Delta},$$

$$\Delta = \tilde{\Delta}_C - \sum_j u_j X_j,$$

$$\Sigma\ (t) = \int_0^t e^{(i\Delta - k)(t-t')}\, \xi\left(t'\right) dt' \,.$$

We note that $\alpha\ (t)$ is an operator valued function of operators X_j which shows the back-action of the condensate on the cavity field. The subsequent bistability has been observed experimentally [7]. Here the resulting expressions for the cavity photon field is formally identical to the single component case. This is not surprising since the cavity is subject only to the combined effect of the two-component condensate. The only non-vanishing second order correlation function for the noise term is

$$\langle \Sigma\left(t\right) \Sigma^{\dagger}\left(t'\right)\rangle = e^{i\Delta(t-t')}\left(e^{-k|t-t'|} - e^{-k(t+t')}\right), \qquad \text{... (5)}$$

which is further approximated by a Lorentzian function

$$\left\langle \Sigma\left(t\right) \Sigma^{\dagger}\left(t'\right)\right\rangle \simeq \frac{2k}{\Delta^2 + k^2}\,\delta\left(t - t'\right). \qquad \text{... (6)}$$

Therefore, the cavity intensity with a noise term can be written as

$$a^{\dagger}a = \left|\alpha\left(t\right)\right|^2 + \Xi\left(t\right),$$

$$\left\langle \Xi\left(t\right) \Xi\left(t'\right)\right\rangle = D\delta\ \left(t - t'\right),$$

$$D = \frac{2k\eta^2}{\left(k^2 + \Delta^2\right)^2}\,. \qquad \text{...(7)}$$

We note that the fluctuation of the cavity field is affected by operator Δ which depends explicitly on operators X_j, again showing the back-action of the condensate. Since the fluctuation of the cavity field is solely related to Δ, our derivations below will resemble that of the single component condensate. As in Ref. [12], we are motivated to perform a perturbative analysis on operators Y_j assuming that the recoil energy is much smaller compared to other energy scales.

We are looking for the amplitude in the following form:

$$\alpha\left(\left\{X_j\right\}, t\right) = \alpha_0\left(\left\{X_j\right\}\right) + \frac{1}{2}\sum_j\left\{Y_j,\, \alpha_1^{(j)}\left(\left\{X_j\right\}\right)\right\}, \qquad \text{... (8)}$$

which is essentially the first order correction to the adiabatic elimination. Here the notation of $\{O_j, O_k\}$ means the anti-commutator of the two enclosed operators O_j and O_k and $\{O_j\}$ stands for the collection of operators. We can see that, to the first order correction, there will be a corresponding correction on Y_j for each component. In its most general form, the coe.cient in front of Y_j depends on both quadratures X_1 and X_2.

With the help of the identity

$$\frac{d\alpha}{dt} = \frac{\partial\alpha}{\partial t} + \sum_j 2\omega_{R,\,j}\left\{Y_j,\, \frac{\partial\alpha}{\partial X_j}\right\}, \qquad \text{... (9)}$$

we obtain terms in the order of Y_j by comparing Eq. (4a) and Eq. (8).

The zeroth order term gives

$$\alpha_0\left(\left\{X_j\right\}\right) = \frac{\eta}{k - i\Delta}, \qquad \text{... (10)}$$

and the first order term gives

$$\alpha_1^{(j)}\left(\left\{X_j\right\}\right) = \frac{4\omega_{R,\,j}}{i\Delta - k}\frac{\partial\alpha_0}{\partial X_j}$$

$$= 4i\omega_{R,\,j}\,\frac{u_j\eta}{\left(k - i\Delta\right)^3}. \qquad \text{... (11)}$$

Finally we express the cavity .eld intensity up to the first order of Y_j

$$a^{\dagger}a = \frac{1}{2}\left(\alpha^*\alpha + \alpha\alpha^*\right)$$

$$= \left|\alpha_0\right|^2 + \frac{1}{2}\sum_j\left\{Y_j,\, \alpha_0^*\alpha_1^{(j)} + \alpha_0\alpha_1^{(j)*}\right\}, \qquad \text{... (12)}$$

from which we can extract the friction force on component j as

$$\mathcal{F}_j = -u_j\left(a^{\dagger}a - \left|\alpha_0\right|^2\right)$$

$$= -\frac{1}{2}u_j\sum_k\left\{Y_k,\, \alpha_0^*\alpha_1^{(k)} + \alpha_0\alpha_1^{(k)*}\right\}$$

$$= -\frac{1}{2}u_j\sum_k\left\{Y_k,\, \Gamma_k\left(\left\{X_j\right\}\right)\right\} \qquad \text{... (13)}$$

with

$$\Gamma_k\left(\{X_j\}\right) = -16\omega_{R,\,k}\,\frac{u_k\eta^2 k\Delta}{\left(k^2 + \Delta^2\right)^3}. \qquad \text{... (14)}$$

We note that both $\mathcal{F}_j$ and $\Gamma_j(\{X_j\})$ are proportional u_j. Thus the intra-species friction force is $\propto u_j^2$ and the inter-species friction force $\propto u_1 u_2$.

After adiabatically removing the cavity field, the reduced equations of motion for the quadratures are given by

$$\begin{aligned} \dot{X}_j &= 4\omega_{R,j} Y_j, \\ \dot{Y}_j &= -4\omega_{R,j} X_j - u_j |\alpha_0|^2 \\ &= -\frac{1}{2} u_j \sum_k \left\{ Y_k, \Gamma_k\left(\left\{X_j\right\}\right) + u_j \Xi \right\} \quad \text{... (15)} \end{aligned}$$

We note that the friction term takes a quadratic form of u_j and the dissipation term a linear form of u_j, which is a manifest of the fluctuation-dissipation theorem.

The master equation corresponding to the above quantum Langevin equations should be in the following form

$$\dot{\rho} = -i\left[H_{eff}, \rho\right] + \mathcal{L}_{fric}\rho + \mathcal{L}_{diff}\,\rho,$$

$$H_{eff} = \sum_j 2\omega_{R,j}\left(X_j^2 + Y_j^2\right) - \frac{\eta^2}{k}\, arc\tan\left(\frac{\Delta}{k}\right),$$

$$\mathcal{L}_{fric}\,\rho = -(i/2)\sum_j \left[g_j\left(\left\{X_j\right\}\right), \left\{Y_j, \rho\right\}\right],$$

$$\mathcal{L}_{diff}\,\rho = -\left[d\left(\left\{X_j\right\}\right), \left[d\left(\left\{X_j\right\}\right), \rho\right]\right] \quad \text{... (16)}$$

where $\mathcal{L}_{fric}$ takes a form similar to the famous Caldeira–Leggett master equation [17].

We need the exact form of $g_j(\{X_j\})$ and $d(\{X_j\})$ to construct the full master equation. To this end, we note that $g_j(\{X_j\})$ is related to the drifting of Y_j which can be determined by matching $\langle Y_j \rangle$ from the master equation and the Heisenberg–Langevian equation. Similarly, $d(\{X_j\})$ is related to the diffusion of Y_j which can be determined by matching $\langle Y_j^2 \rangle$.

For the friction term ($k \neq j$), we have

$$\frac{\partial_{gk}}{\partial X_j} = u_j \Gamma_k = -16\omega_{R,j} \frac{u_j u_k \eta^2 k \Delta}{\left(k^2 + \Delta^2\right)^3} \quad \text{... (17)}$$

from which we obtain

$$g_k(\{X_j\}) = -4\omega_{R,k} \frac{u_k k \eta^2}{\left(k^2 + \Delta^2\right)^2}. \quad \text{... (18)}$$

For the diffusion term, we have

$$2\left(\frac{\partial d}{\partial X_j}\right)^2 = u_j^2 D\left(\left\{X_j\right\}\right) \quad \text{... (19)}$$

from which we obtain

$$d(\{X_j\}) = \frac{\eta}{\sqrt{k}}\, arc\tan\left(\frac{\Delta}{k}\right). \quad \text{... (20)}$$

In general, the dynamics for the above master equation is computationally difficult to solve. In the following, we concentrate on a specific case with $\omega_{R,1} = \omega_{R,2} = \omega_R$, i.e. the two components refer to two different internal states for the same alkali (there is no conversion between them since $\Omega_0 = 0$). u_j can be changed by using different atom numbers N_j. Denoting $g_k(\{X_j\}) = u_k \tilde{G}$, we have

$$\tilde{G} = -4\frac{\omega_R k \eta^2}{\left(k^2 + \Delta^2\right)^2}, \quad \text{... (21)}$$

independent of atomic species. The friction term is $\mathcal{L}_{fric}\,\rho = -(i/2)\sum_j \left[\tilde{G}\left(\left\{X_j\right\}\right), u_j\left\{Y_j, \rho\right\}\right]$.

By making coordinate transformations ($u^2 = u_1^2 + u_2^2$)

$$X_1 = \frac{u_1 X + u_2 x}{u}, \qquad X_2 = \frac{u_2 X - u_1 x}{u},$$

$$Y_1 = \frac{u_1 Y - u_2 y}{u}, \qquad Y_2 = \frac{u_2 Y - u_1 y}{u}, \quad \text{... (22)}$$

we can rewrite the master equation as

$$\dot{\rho} = -i\left[H_{eff}, \rho\right] + \mathcal{L}_{fric}\,\rho + \mathcal{L}_{diff}\,\rho$$

$$H_{eff} = 2\omega_R (X^2 + Y^2 + x^2 + y^2) - \frac{\eta^2}{k}\, arc\tan\left(\frac{\tilde{\Delta}_c - uX}{k}\right)$$

$$\mathcal{L}_{fric}\,\rho = -(i/2)\sum_j \left[G(X), \{Y, \rho\}\right]$$

$$G(X) = -4\frac{\omega_R u k \eta^2}{\left(k^2 + \left(\tilde{\Delta}_c - uX\right)^2\right)^2}$$

$$\mathcal{L}_{diff}\ \rho = -\left[d\left(X\right),\left[d\left(X\right),\rho\right]\right]$$

$$d\ (X) = \frac{\eta}{\sqrt{k}}\ arc\ \tan\left(\frac{\tilde{\Delta}_c - uX}{k}\right) \qquad \text{... (23)}$$

We can immediately see that the master equation is separable in the center of mass coordinates (X, Y) and the relative coordinates (x, y). This allows us to treat them separately. Here u_j plays the role of effective mass in the coordinate transformations. Note that u_j is not proportional to N_j but $\sqrt{N_j}$. The radiation pressure from the cavity acts only on the center of mass motion of the two-component condensate and has no influence on the corresponding relative motion. We emphasize that the decoupling of the center of mass and relative motion is also clear from Eqs (4a)-(4c) where the relative motion is decoupled from the center of mass motion under the transformation (Eq. 22). Both the friction term and the noise term have been cancelled out in the relative motion.

We assume the initial density matrix separable in the center of mass and relative motion, i.e. $\rho\ (t = 0) = \rho_s\ (X, X', t = 0)\ \rho_o\ (x, x')$ with $\rho_o\ (x, x') = \phi_0\ (x)\ \phi_0\ (x')$ where $\phi_0\ (x)$ is the lowest eigenstate of the harmonic oscillator, $2\ (x^2 + y^2)$. Therefore, at any time t, $\rho_s\ (X, X', t)$ satisfies

$$\dot{\rho}_s = -i\ [H_{eff,\ s}, \rho_s] + \mathcal{L}_{fric}\rho_s + \mathcal{L}_{\text{diff}}\ \rho_s \qquad \text{... (24)}$$

$$H_{eff},\ _s = 2\omega_R\ (X^2 + Y^2) - \frac{\eta^2}{k}\ arc\ \tan\left(\frac{\tilde{\Delta}_c - uX}{k}\right)$$

By reducing the original two-component problem to a one-body problem, we can numerically integrate the master equation (Eq. 24) and compute the dynamics for various operators. Because of the large parameter space, we do not want to explore every possible cases. Instead, we focus on a representative case which is also discussed in Ref. [12]: $k = 342.1\ \omega_R$, $\tilde{\Delta}_c = -825\omega_R$, $u = 169.74\ \omega_R = \sqrt{2}u_1 = \sqrt{2}u_2$, and $\eta = 120\ \omega_R$. We will consistently use these parameters unless otherwise specified. For these parameter values, it turns out that the potential in $H_{eff,\ s}$ exhibits a double well structure which falls into the so-called bistability regime. Such a regime has been discussed in detail [12] and we simply quote their results here.

The initial state for ρ_s is chosen as $\rho_s\ (t = 0) = |\psi\ (t = 0)\rangle\ \langle\psi\ (t = 0)|$ with $\langle X|\psi\ (t = 0)\rangle = 1/\sqrt{2}\ (\psi_0\ (X) + \psi_1\ (X))$. $\psi_1\ (X)$). ψ_0, 1 (X) are the two lowest eigenstates of $H_{\text{eff},\ s}$.

Due to the double well structure, the initial state $\psi\ (X)$ is mostly localized in the left potential well. Without dissipation and friction, there will be Josephson oscillations between the two potential wells. The existence of diffusion and friction alters the dynamics significantly.

To demonstrate the difference, we numerically solve the master equation and our results are shown below. We first compute $\langle X_1\rangle$ as function of time t. From the transformation of Eq. (22), we have $\langle X_j\rangle = \frac{u_j}{u}\langle X\rangle$. Therefore, X_1 and X_2 show the same behavior as in the single component case. This result is easy to understand since the radiation pressure always drives the two components in phase. More interestingly, the amplitude of $\langle X_j\rangle$ is proportional to u_j which implies that the average displacement depends linearly on the square root of atom number. Recall in this study we have neglected the s-wave scattering potential, this dependence on atom number is purely a back action of the cavity field and differs from the usual nonlinearity with the mean field approximation. Our numerical results are shown in Fig. 1. We can clearly see damped oscillations due to diffusion and friction since there will be coherent oscillation between the two wells if there are no friction and dissipation. Similar results are also reported in Ref. [12]. The decay rate is found to be about 10 times larger than the Josephson tunnelling rate so that Josephson oscilla-tions are strongly suppressed [12]. As a consequence, the intra-cavity field intensity $\langle a^\dagger a\rangle$ is expected to show a similar damped behavior. Such a damping will reduce the average intracavity photon. From Eq. (12) and the transformation of Eq. (22), we have

$$\left\langle a^\dagger a\right\rangle = \left\langle |\alpha_0|^2 + \left\{Y, \frac{-8\omega_R k\eta^2\ u\left(\tilde{\Delta}_c - uX\right)}{\left(k^2 + \left(\tilde{\Delta}_c - uX\right)^2\right)^3}\right\}\right\rangle. \qquad \text{... (25)}$$

The first term is the leading term where it already takes the back action of condensed atoms into consi-deration. The second term is merely a first order correction and does not play important role for the chosen parameters. The numerical results are shown in Fig. 2. It is easy to observe that there is roughly a π phase between $\langle X_1\rangle$ and $\langle a^\dagger a\rangle$. This can be seen from Eq. (10) where α_0 is inversely related to Δ. So a local maximum in α_0

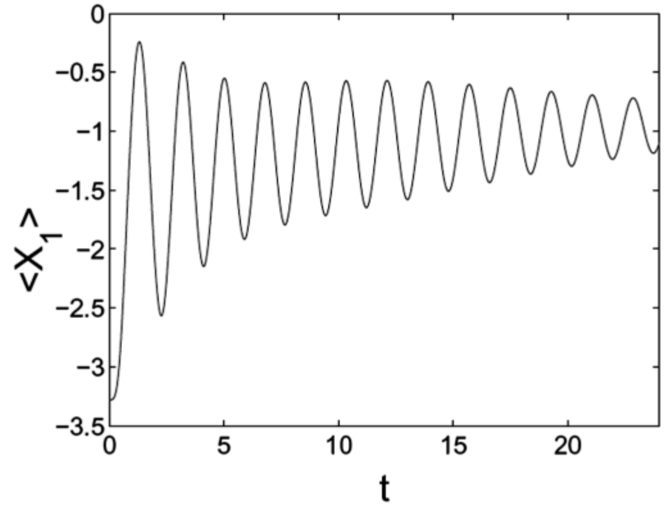

Fig. 1: $\langle X_1 \rangle$ as function of time t. Time is in unit of $1/\omega_R$.

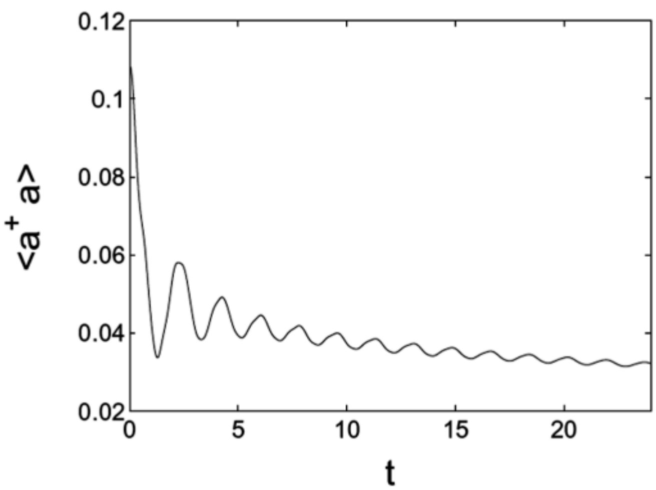

Fig. 2: $\langle a^\dagger a \rangle$ as function of time t. Time is in unit of $1/\omega_R$.

corresponds to a local minimum in $\langle X_1 \rangle$ and vice verse. However, both $\langle X_1 \rangle$ and $\langle a^\dagger a \rangle$ saturate to finite values.

For friction and diffusion process, mixedness is another important quantity that deserves some discussions. It is defined as $S = 1 - Tr\ (\rho^2) = 1 - Tr\ (\rho_s^2)$. Mixedness characterizes the loss of coherence in a quantum system. Without diffusion and friction, ρ is always associated with a single pure state and S will always be 0. However, this is not the case when we include diffusion and friction as can be seen from our numerical results in Fig. 3. It turns out that there are two major differences compared to $\langle X_1 \rangle$ and $\langle a^\dagger a \rangle$. First, we find that S is a monotonically increasing function of time t despite the oscillations in $\langle X_1 \rangle$ and $\langle a^\dagger a \rangle$. Second, the mixedness of our state increases at a time scale of ~ 0.1 which is much

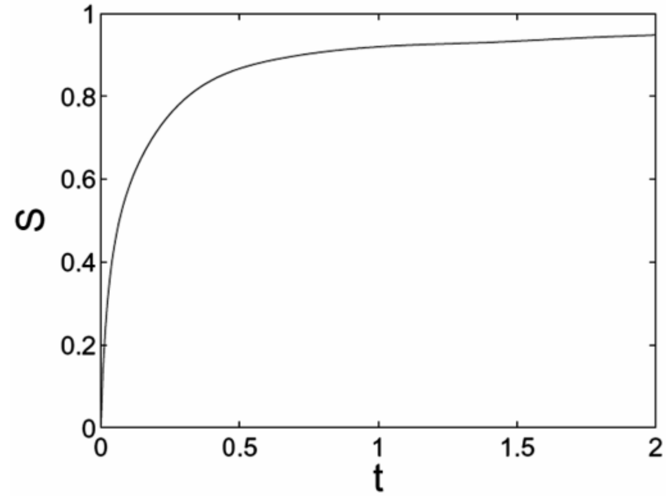

Fig. 3: Linear entropy S (mixedness) as function of time t. Time is in unit of $1/\omega_R$.

smaller than the oscillation period $\langle X_1 \rangle$ and $\langle a^\dagger a \rangle$. Denoting the time scale of Josephson oscillation, damped oscillation, and mixedness as τ_J, τ_D, and τ_M, respectively, they satisfy $\tau_J \sim 6\tau_D$ and τ_D » τ_M in the current model. This clearly shows that the di.usion term plays a dominant role.

For a two-component condensate, we are particularly interested in the correlation dynamics between different quadratures under a common radiation field, e.g. X_1 and X_2, X_1 and Y_2. While such correlation effect is absent in a single component condensate, it naturally arises in the two component case. To this end, we define the normalized covariance for two operators P and Q to characterize the degree of correlation

$$C\,(P, Q) = \frac{\langle PQ \rangle + \langle QP \rangle - 2\,\langle P \rangle \langle Q \rangle}{2\,\sqrt{\Delta P^2 \Delta Q^2}}\,.$$

If $P = Q$, $C\,(P, Q) = 1$. If P and Q are not correlated, $C\,(P, Q) = 0$. Therefore, $C\,(P, Q)$ gives an overall feature of correlation. From the transformation of Eq. (22), we can obtain various expressions for $C\,(X_j, X_k)$ and $C\,(X_j, Y_k)$. For example, we have

$$C\left(X_1, X_2\right) = \frac{\frac{u_1 u_2}{u^2}\left(\Delta X^2 - \Delta x^2\right)}{\prod_j \left(\frac{u_j^2}{u^2}\left(\Delta X^2 - \Delta x^2\right) + \Delta x^2\right)^{\frac{1}{2}}} \quad \text{... (26)}$$

Our numerical results for $C\,(X_1, X_2)$ as function of time t are shown in Fig. 4. Other $C\,(X_j, Y_k)$ shows similar

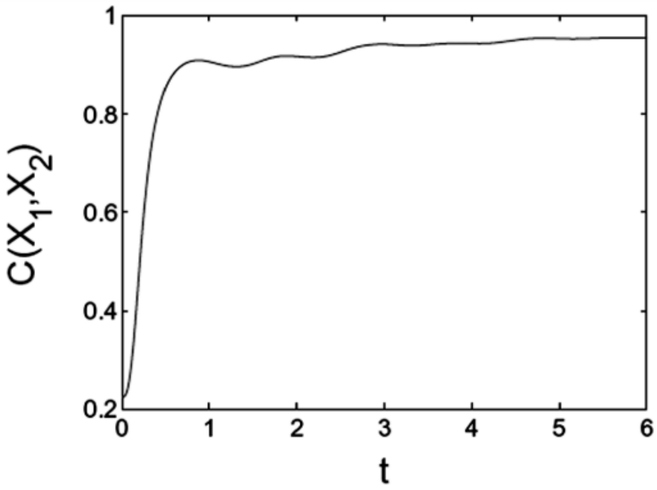

Fig. 4: Covariance C (X_1, X_2) as Function of time t. Time is in unit of $1/\ \omega_R$

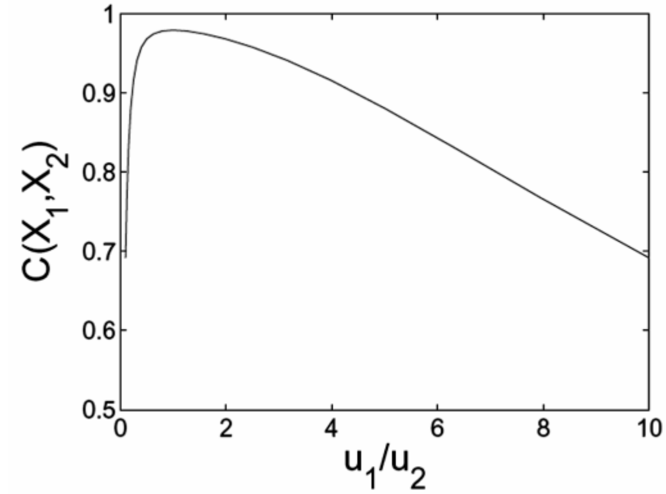

Fig. 5: Covariance C (X_1, X_2) as Function of u_1/u_2 for Fixed $u = 169.74\omega_R$ at $t = 24/\omega_R$.

patterns and will not be addressed any more. From Fig. 4, we can see that, in spite of a few small amplitude oscillations at the initial stage, the overall trend of C (X_1, X_2) is to increase and saturate to a constant value. While the time scale associated with the increase stage is similar to that of mixedness, the time scale of small amplitude oscillations is about the same as that in $\langle X_1 \rangle$ and $\langle a^\dagger a \rangle$. The damping effect due to diffusion and friction dominates the Josephson tunnelling effect in the condensate dynamics. It may be surprising that correlation between X_1 and X_2 may increase to such an extent in the presence of dissipation and diffusion. Analysis show that the wave packet actually becomes more flatten due to dissipation and diffusion. Such a phenomenon is a direct result of the condensate motion driven by a common cavity field. It is analogous to the so-called environment-assisted entanglement [18].

To further show the dependence of C (X_1, X_2) on u_1/u_2, we present our numerical results in Fig. 5 with fixed $u = 169.74\,\omega_R$ at $t = 24/\omega_R$. We find the peak is achieved for $u_1 = u_2$, i.e. the two quadratures are symmetrically coupled to the cavity field. This can be easily understood since the two quadratures are coupled to the cavity field in the same fashion and u_j is the only parameter that is associated with quadrature X_j.

In conclusion, we have shown the dynamics for a two component Bose-Einstein condensate which is dispersively coupled to a high-Q optical cavity. By adiabatically eliminating the cavity field, we obtain a master equation for the quadratures of a two-component condensate. As a result of diffusion and friction, the dynamics exhibits a strong damping effect which is different from the non-dissipative case. We give our numerical results for the average of quadratures and the intensity of the intra-cavity field as a function of time. We also present the time dependence of linear entropy and normalized covariance between quadratures of the condensate. All of these demonstrate that damping plays an important role in the dynamics for a two-component condensate. For a future study, we plan to investigate the effect due to transition between the two components. We hope our results reported here can be helpful to ongoing experimental efforts in exploring the combined condensate-cavity system.

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