# On Asymptotic Lacunary Statistically Equivalent Sequences in Intuitionistic Fuzzy Normed Space

**Shailendra Pandit[1], Ayaz Ahmad[2]**

[1] M.Sc from Indian Institute of technology Bhubaneswar (India)

Ph.D at National Institute of technology Patna (India). Since 2019

[2]M.Sc from Patna University Patna

Ph.D from Aligarh Muslim University Aligarh.

**Abstract**

In this article, we have studied asymptotic equivalent, asymptotic statistically equivalent, strongly asymptotic statistically equivalent, asymptotic lacunarry statistically equivalent, and strongly asymptotic lacunary statistically equivalent of multiple λ with respect to Intuitionistic fuzzy normed space. In addition, we have proved some theorems on asymptotic lacunary equivalent and strongly asymptotic lacunary equivalent sequences. Furthermore, refinement of lacunary sequence has been defined and some results over it were examined.

**Keywords:** Asymptotic equivalence, Lacunary sequence, Lacunary refinement, *θ−density*, Intuitionistic fuzzy normed space.

## 1 Introduction

The notion of statistical convergence has been given by both Fast [1] in 1951 and Fridy[3] in 1985, independently. which emerged as a useful tool in dealing with such numerical problems where the idea of ordinary convergence fails to serve the purpose of study, by the idea of the natural density. Later, the concept was studied by several authors in their own ways and was generalized it to some different kinds of spaces. Kostyrko [14] generalised the statistical convergence to ideal convergence, Salat and Tripathy[15] studied some properties of ideal convergence. Further Das and Kostyrko [16] investigate the notion of I-convergence and I∗-convergence. Later, the concept of lacunary statistical convergence was investigated by Friday and Orhan[4] in 1993. Since then, several authors have studied this concept and extended it to different spaces. Some authors [5, 8] studied the idea in intuitionistic fuzzy normed spaces, which was first studied by Saadati and Park[18] as a generalization of fuzzy metric space in 2006. The Fuzzy metric space was studied by Karmosil[17] as a generalised form of fuzzy set theory, which is being used not only in different branches of mathematics, such as the theory of functions [10, 11], theory of approximation [12] etc, but also in quantum particle physics[13].

A number of developments in statistical convergence and its generalisation to different spaces have been made by different authors in recent years. Patterson [6] evaluated some results on asymptotically statistically equivalent sequences in 2003. Patterson and Savas[7] again presented their work on asymptotically lacunary in 2006 statistically equivalent sequences. The same idea was studied and extended by Ulusu and F. Nuray [9] to the set of sequences (in Wijsman's sense). They published their work on asymptotically lacunary statistical equivalent set sequences in 2013.

We have now investigated the asymptotic statistically equivalence sequences, asymptotic lacunary equivalence sequences, and strong asymptotic lacunary equivalent sequences in intuitionistic fuzzy normed space. Throughout this article, X stands for linear space, N stands for the set of natural integers, and λ is a real number.

The work is divided into three sections. Section 1 contains an introduction part, Section 2 contains the basic definitions and results that are needed in the work. Section 3 contains the main definitions and results, which we have studied

## 2. Basics and Preliminaries

**Definition 2.1.** [5] A 5-tuple object $(X, \alpha, \beta, *, o)$ where X is linear space, $*$ is continuous t-norm, o is continuous t-conorm and $\alpha, \beta$ are fuzzy sets on $X\times(0, \infty)$, is said to be Intuitionistic fuzy normed space (abbreviated as IFNS) if for every $x, y \in X$ and $\mu, \xi \in (0, \infty)$ it fulfils the following subsequent requirements.

(i) $0<\alpha(x, \mu)+\beta(x, \mu)\le 1$

(ii) $\alpha(x, \mu) \geq 0$

(iii) $\alpha(cx, \mu) = \alpha\left(x, \frac{\mu}{|c|}\right)$ for all $c \neq 0$

(iv) $\alpha(x, \mu) = 1$ iff $x = 0$

(v) $\alpha(x + x_1, \mu + \xi) \geq \alpha(x, \mu) * \alpha(x_1, \xi)$

(vi) $\alpha(x, \mu) : (0, \infty) \to [0, 1]$ is a continuous function of $\mu$

(vii) $\lim_{\mu\to\infty} \alpha(x, \mu) = 1$ and $\lim_{\mu\to 0} \alpha(x, \mu) = 0$

(viii) $\beta(x, \mu) < 1$

(ix) $\beta(cx, \mu) = \alpha\left(x, \frac{\mu}{|c|}\right)$ for all $c \neq 0$

(x) $\beta(x, \mu) = 1$ iff $x = 0$

(xi) $\beta(x + x_1, \mu + \xi) \leq \alpha(x, \mu) o \alpha(x_1, \xi)$

(xii) $\beta(x, \mu) : (0, \infty) \to [0, 1]$ is a continuous function of $\mu$

(xiii) $\lim_{\mu\to\infty} \beta(x, \mu) = 0$ and $\lim_{\mu\to 0} \beta(x, \mu) = 1$

The doublet $(\alpha, \beta)$ defines Inuitionistic fuzzy norm (abbreviated as IFN) and given by, $(\alpha, \beta) = \{(x, y) : \alpha(x, \mu), \beta(x, \mu) : x \in X\}$

**Definition 2.2.** A sequence $[x] = (x_r)$ of elements of IFNS $(X, \alpha, \beta, *, o)$ is said to be converges to a number *l*, if there exist $r_0 \in N$ such that for all $r \geq r_0$, and $\mu \in (0, \infty)$ we have $1 - \alpha(x_r - l, \mu) < \varepsilon$ and $\beta(x_r - l, \mu) < \varepsilon$ for every $0 < \varepsilon < 1$

**Definition 2.3.** Two sequences $[x] = (x_r)$ and $[y] = (y_r)$ of non-negative terms of IFNS $(X, \alpha, \beta, *, o)$ are said to be asymptotically equivalent of multiple $\lambda$ with respect to IFN $(\alpha, \beta)$ if there exist some positive integer $r_0$ such that for all $0 < \varepsilon < 1$ and $\mu > 0$ we have, $1 - \alpha\left(\frac{x_r}{y_r} - \lambda, \mu\right) < \varepsilon$ and $\beta\left(\frac{x_r}{y_r} - \lambda, \mu\right) < \varepsilon$, for all $r \geq r_0$

Symbollically, we state it by, $[x] \sim [y]$.

The two sequences are said to be simply asymptotically equivalent if $\lambda = 1$

**Definition 2.4.** Two sequences $[x] = (x_r)$ and $[y] = (y_r)$ of non-negative terms of IFNS $(X, \alpha, \beta, *, o)$ are said to be asymptotic statically equivalent of multiple $\lambda$ with respect to IFN $(\alpha, \beta)$ if for all $0 < \varepsilon < 1$ and $\mu > 0$ we have,

$$\delta\left\{k \in \mathbb{N} : 1 - \alpha\left(\frac{x_k}{y_k} - \lambda, \mu\right) \geq \varepsilon \text{ or } \beta\left(\frac{x_k}{y_k} - \lambda, \mu\right) \geq \varepsilon\right\} = 0$$

Symbolically, we state it by, $[x] \overset{S_\lambda}{\sim} [y]$.

The two sequences are said to be simply asymptotic statistically equivalent if $\lambda = 1$

**Remark 1.** $\delta(A)$ denotes the natural density of set *A* (See [3]), defined by $\delta(A)=\lim_{n\to\infty}\frac{|A|}{n}$ where $|A|$ gives the cardinality of the set *A*

**Definition 2.5** [9] A lacunary sequence $\theta=(u_k)$ is sequence of non-negative integers such that $u_0=0,\ h_k=u_k-u_{k-1}\to\infty$ as $k\to\infty$, throughout the paper we denote $q_k=\frac{u_k}{u_{k-1}}$ and $I_k=(u_{k-1},\ u_k]$

**Definition 2.6.** [2] A lacunary sequence $\sigma=\{v_r\}$ of non-negative integers is called refinement of lacunary sequence $\theta=\{u_r\}$ if $\{u_r\}\subseteq\{v_r\}$

**Definition 2.7** [8] A sequence $(x_r)$ of the terms of IFNS $(X,\ \alpha,\ \beta,\ *,\ o)$ said to be lacunary statistically convergent to *l* with respect to IFN $(\alpha,\ \beta)$ if for all $0<\varepsilon<1$ and $\mu>0$ we have, $\delta_\theta\{k:1-\alpha(x_k-l,\ \mu)\geq\varepsilon\ or\ \beta(x_k-l,\ \mu)\geq\varepsilon\}=0$

## 3. Main Results

**Definition 3.1.** Two sequences $[x]=(x_r)$ and $[y]=(y_r)$ of non-negative terms of IFNS $(X,\ \alpha,\ \beta,\ *,\ o)$ are said to be asymptotic lacunary statistically equivalent of multiple $\lambda$ with respect to IFN $(\alpha,\ \beta)$ if for all $0<\varepsilon<1$ and $\mu>0$ we have,

$$\delta_\theta\left\{k:1-\alpha\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)\geq\varepsilon\ or\ \beta\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)\geq\varepsilon\right\}=0$$

Symbollically, we state it by, $(\alpha,\beta)-\ [x]\overset{S_\theta^\lambda}{\sim}[y]$.

The two sequences are said to be simply asymptotic lacunary statistically equivalent if $\lambda=1$

**Remark 2.** $\delta_\theta(A)$ denotes the lacunary natural density $(\theta-\text{density})$ of set A (See [4]), defined by $\delta_\theta(A)=\lim_{k\to\infty}|\{q\in N:q\in I_k\cap A\}|$ where vertical bar gives the cardinality of the set *A*

**Definition 3.2.** Two sequences $[x]=(x_r)$ and $[y]=(y_r)$ of non-negative terms of IFNS $(X,\ \alpha,\ \beta,\ *,\ o)$ are said to be strongly asymptotic statistically equivalent of multiple $\lambda$ with respect to IFN $(\alpha,\ \beta)$ if for all $0<\varepsilon<1$ and $\mu>0$ we have, $\lim_{n\to\infty}\frac{1}{n}\sum_{k\geq1}^{n}\alpha\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)=1$ and $\lim_{n\to\infty}\frac{1}{n}\sum_{k\geq1}^{n}\beta\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)=0.$ Symbolically, we state it by, $(\alpha,\beta)-[x]\overset{N_\lambda}{\sim}[y]$.

The two sequences are said to be simply asymptotic lacunary statistically equivalent if $\lambda=1$

**Example 3.1.** Let the two sequence [*x*] and [*y*] are defined on IFNS $(\square,\ \alpha,\ \beta,\ *,\ o)$, defined by $[x]=(x_k)=$ Area of a circle of radius *k* unit and $[y]=(y_k)=k^2e^{\frac{1}{k}}$. Let us define $\alpha\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)=\frac{\mu-2^{-k}}{\mu+\left|\frac{x_k}{y_k}-\lambda\right|}$ and $\beta\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)=\frac{\left|\frac{x_k}{y_k}-\lambda\right|}{\mu+\left|\frac{x_k}{y_k}-\lambda\right|}$.

Also, $a*b=\min\{a,\ b\}$ and $aob=\max\{a,\ b\}$. Then we have $(\alpha,\beta)-[x]\overset{N_\pi}{\sim}[y]$.

*Proof.* $\lim_{n\to\infty}\sum_{k\geq1}^{n}\alpha\left(\frac{x_k}{y_k}-\pi,\ \mu\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{k\geq1}\frac{\mu-2^{-k}}{\mu+\left|\pi e^{-\frac{1}{k}}-\pi\right|}=\lim_{n\to\infty}\frac{1}{n}\sum_{k\geq1}\frac{\mu-2^{-k}}{\mu+\pi\left|e^{-\frac{1}{k}}-1\right|}$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{k\geq 1}^{k_0}\frac{\mu-2^{-k}}{\mu+\pi\left|e^{-\frac{1}{k}}-1\right|}+\lim_{n\to\infty}\frac{1}{n}\sum_{k\geq k_0}^{n}\frac{\mu-2^{-k}}{\mu+\pi\left|e^{-\frac{1}{k}}-1\right|}=\lim_{n\to\infty}\frac{1}{n}\sum_{k\geq 1}^{k_0}(\text{finite})+$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k\geq k_0}^{n}(n-k_0)=0+\lim_{n\to\infty}\frac{n-k_0}{n}=1$$

(Since we have a positive integer $k_0$ such that $\frac{1}{2^k}\approx 0$ and $e^{-\frac{1}{k}}\approx 1$ for all $k\geq k_0$)

Also,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k\geq 1}^{n}\beta\left(\frac{x_k}{y_k}-\pi,\ \mu\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\pi\left|e^{-\frac{1}{k}}-1\right|}{\mu+\pi\left|e^{-\frac{1}{k}}-1\right|}=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{k_0}\frac{\pi\left|e^{-\frac{1}{k}}-1\right|}{\mu+\pi\left|e^{-\frac{1}{k}}-1\right|}+$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=k_0}^{n}\frac{\pi\left|e^{-\frac{1}{k}}-1\right|}{\mu+\pi\left|e^{-\frac{1}{k}}-1\right|}=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{k_0}\text{finite}+0.=0$$

**Definition 3.3.** Let $\theta=(u_r)$ be a lacunary sequence. The two sequences $x=[x_r]$ and $y=[y_r]$ of non-negative terms of IFNS $(X,\ \alpha,\ \beta,\ *,\ o)$ are said to be strongly asymptotic lacunary satistically equivalent of multiple $\lambda$ with respect to IFN $(\alpha,\ \beta)$ if for all $0<\varepsilon<1$ and $\mu>0$ we have, $\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\alpha\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)=1$ and $\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\beta\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)=0$

Symbolically, we denote it by, $(\alpha,\beta)-\ [x]\overset{N_\theta^\lambda}{\sim}[y]$.

The two sequence are said to be simply asymptotic lacunary equivalent if $\lambda=1$

**Example 3.2.** Let the two sequence $[x]$ and $[y]$ are defined on IFNS $(\square,\ \alpha,\ \beta,\ *,\ o)$, defined by

$$[x]=x_k=\begin{cases}k^3: & u_{r-1}<k\leq u_{r-1}+\left[\sqrt{h_r}\right]\\ \pi k^2: & \text{otherwise}\end{cases}$$

and $[y]=y_k=k^2 \quad : \qquad u_{r-1}<k\leq u_r$

Where $\left[\sqrt{h_r}\right]$ denotes greatest integral value below $\sqrt{h_r}$. Define $\alpha(x-y,\ \mu)=\frac{\mu}{\mu+|x-y|}$ and $\beta(x-y,\ \mu)=\frac{|x-y|}{\mu+|x-y|}$. Also, $a*b=\min\{a,\ b\}$ and $aob=\max\{a,\ b\}$

Then we have $(\alpha,\beta)-[x]\overset{N_\theta^\pi}{\sim}[y]$.

*Proof.* $\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\alpha\left(\frac{x_k}{y_k}-\pi,\ \mu\right)=\lim_{r\to\infty}\sum_{u_{r-1}<k\leq u_{r-1}+\left[\sqrt{h_r}\right]}\alpha\left(\frac{x_k}{y_k}-\pi,\ \mu\right)+$

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{u_{r-1}+\left[\sqrt{h_r}\right]<k\le u_r}\alpha\left(\frac{x_k}{y_k}-\pi,\ \mu\right)=\lim_{n\to\infty}\frac{1}{h_r}\sum_{u_{r-1}<k\le u_{r-1}+\left[\sqrt{h_r}\right]}\frac{\mu}{\mu+\left|\frac{k^3}{k^2}-\pi\right|}+$$

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{u_{r-1}+\left[\sqrt{h_r}\right]<k\le u_r}\frac{\mu}{\mu+\left|\frac{\pi k^2}{k^2}-\pi\right|}\le\lim_{r\to\infty}\frac{\left[\sqrt{h_r}\right]}{h_r}+\lim_{r\to\infty}\frac{u_r-u_{r-1}-\left[\sqrt{h_r}\right]}{h_r}$$

$$=\lim_{r\to\infty}\frac{\left[\sqrt{h_r}\right]}{h_r}+\lim_{r\to\infty}\left(1-\frac{\left[\sqrt{h_r}\right]}{h_r}\right)=1.$$

Also,

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\beta\left(\frac{x_k}{y_k}-\pi,\ \mu\right)=\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\left[1-\alpha\left(\frac{x_k}{y_k}-\pi,\ \mu\right)\right]=$$

$$1-\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\alpha\left(\frac{x_k}{y_k}-\pi,\ \mu\right)=1-1=0.$$ Which gives $(\alpha,\beta)-[x]\overset{N_\theta^\pi}{\sim}[y]$.

**Lemma 3.4.** If $\theta=(u_r)$ be a lacunary sequence, $[x]=(x_r)$ and $[y]=(y_r)$ are two sequences of non-negative terms of IFNS $(X,\ \alpha,\ \beta,\ *,\ o)$ then following are equivalent.

(i) $(\alpha,\beta)-[x]\overset{N_\theta^\pi}{\sim}[y]$.

(ii) $\delta_\theta\left\{k\in\mathbb{N}:1-\alpha\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)\ge\varepsilon\right\}=\delta_\theta\left\{k\in\mathbb{N}:\beta\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)\ge\varepsilon\right\}=0.$

(iii) $\delta_\theta\left\{k\in\mathbb{N}:1-\alpha\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)<\varepsilon\right\}=\delta_\theta\left\{k\in\mathbb{N}:\beta\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)<\varepsilon\right\}=1.$

**Theorem 3.5.** $\theta=(u_r)$ be a lacunary sequence. $[x]=(x_r)$ and $[y]=(y_r)$ are two sequences of non-negative terms of IFNS $(X,\ \alpha,\beta,\ *,\ o)$ then following are equivalent.

(i) $(\alpha,\beta)-[x]\overset{N_\theta^\lambda}{\sim}[y]$.

(ii) $(\alpha,\beta)-[x]\overset{S_\theta^\lambda}{\sim}[y]$.

*Proof.* (i) $\to$(ii)

$(\alpha,\beta)-[x]\overset{N_\theta^\lambda}{\sim}[y]$. For every $0<\varepsilon<1$ and $\mu>0$ we have

$$\lim_{k\to\infty}\frac{1}{h_k}\sum_{p\in I_k}\alpha\left(\frac{x_p}{y_p}-\lambda,\ \mu\right)=1 \text{ and } \lim_{k\to\infty}\frac{1}{h_k}\sum_{p\in I_k}\beta\left(\frac{x_p}{y_p}-\lambda,\ \mu\right)=0$$

$$\Rightarrow\lim_{k\to\infty}\frac{1}{h_k}\sum_{p\in I_k}\left\{1-\alpha\left(\frac{x_p}{y_p}-\lambda,\ \mu\right)\right\}=0$$

Now we have

$$\Rightarrow \lim_{k\to\infty}\frac{1}{h_k}\sum_{p\in I_k}\left\{1-\alpha\left(\frac{x_p}{y_p}-\lambda,\ \mu\right)\right\}\geq \lim_{k\to\infty}\frac{1}{h_k}\sum_{\alpha\left(\frac{x_p}{y_p}-\lambda,\ \mu\right)\leq 1-\epsilon}\left\{1-\alpha\left(\frac{x_p}{y_p}-\lambda,\ \mu\right)\right\}$$

$$\geq \lim_{k\to\infty}\frac{\varepsilon}{h_k}\left|\left\{k:1-\alpha\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)\geq\varepsilon\ or\ \beta\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)\geq\varepsilon\right\}\right|$$

$$\Rightarrow \delta_\theta\left\{k:1-\alpha\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)\geq\varepsilon\ or\ \beta\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)\geq\varepsilon\right\}=0$$

Now, (ii) $\to$ (i)

$(\alpha,\beta)-[x]\overset{S_\theta^\lambda}{\sim}[y]$. For every $0<\epsilon<1$ and $\mu>0$ we have

$$\delta_\theta\left\{k:1-\alpha\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)\geq\varepsilon\ or\ B\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)\geq\varepsilon\right\}=0$$

Now

$$\frac{1}{h_k}\sum_{p\in I_k}\left\{1-\alpha\left(\frac{x_p}{y_p}-\lambda,\ \mu\right)\right\}=\frac{1}{h_k}\sum_{\alpha\left(\frac{x_p}{y_p}-\lambda,\ \mu\right)\leq 1-\varepsilon}\left\{1-\alpha\left(\frac{x_p}{y_p}-\lambda,\ \mu\right)\right\}+$$

$$\frac{1}{h_k}\sum_{\alpha\left(\frac{x_p}{y_p}-\lambda,\ \mu\right)>1-\varepsilon}\left\{1-\alpha\left(\frac{x_p}{y_p}-\lambda,\ \mu\right)\right\}\leq\frac{1}{h_k}\left|\left\{k:1-\alpha\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)\geq\varepsilon\right\}\right|+\varepsilon$$

By lemma 3.1, we get

$$\lim_{k\to\infty}\frac{1}{h_k}\sum_{p\in I_k}\left\{1-\alpha\left(\frac{x_p}{y_p}-\lambda,\ \mu\right)\right\}\leq\lim_{k\to\infty}\frac{1}{h_k}\left|\left\{k:1-\alpha\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)\geq\varepsilon\ or\ \beta\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)\geq\varepsilon\right\}\right|+\varepsilon$$

Since $\varepsilon$ was arbitrary, thus theorem is established.

**Theorem 3.6.** Let $\theta=\{u_r\}$ is a lacunary sequence and $\sigma=\{v_r\}$, be refinement of $\theta=\{v_r\}$, if $[x]=(x_r)$ and $[y]=(y_r)$ are two sequences defined on IFNS $(X,\ \alpha,\ \beta,\ *,\ o)$ then

$$(\alpha,\beta)-\ [x]\overset{S_\sigma^\lambda}{\sim}[y].\ \Longrightarrow(\alpha,\beta)-\ [x]\overset{S_\theta^\lambda}{\sim}[y].$$

*Proof.* Let $(\alpha,\beta)-[x]\overset{S_\theta^\lambda}{\sim}[y]$ holds, thus we have

$$S_\sigma\left\{k\in\mathbb{N}\ :1-\alpha\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)\geq\varepsilon\ or\ \beta\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)\geq\varepsilon\right\}=0.$$

$\lim_{r\to\infty}\frac{1}{h_r^{'}}\left|\left\{k\in I_r^{'}:1-\alpha\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)\geq\varepsilon\ or\ \beta\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)\geq\varepsilon\right\}\right|=0.$ Where $h_r^{'}=v_r-v_{r-1}$ and $I_r^{'}=(v_{r-1},\ v_r]$. Now let $I_r$ of $\theta$ contains the interval $I_{r,i}^{'}$ $i=1,\ 2,\ 3,\ .....,\ n$ such that $I_r=\bigcup_{i=1}^{n}I_{r,i}^{'}$ where $I_{r,1}^{'}=(u_{r-1,0}\ u_{r,1}],\ I_{r,2}^{'}=(u_{r,1}\ u_{r,2}],\ I_{r,n}^{'}=(u_{r,n-1}\ u_{r,n}],\ u_{r-1,0}=u_{r-1},$ and $u_{r-1,n}=u_r$ $u_{r,1}$ is first integer after $u_{r-1}$ in the interval $I_r$ such that $I_{r,1}^{'}$ forms a first block for $\sigma=\{v_r\}$ in $I_r$ of $\theta$. Now

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r:1-\alpha\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)\geq\varepsilon\ or\ \beta\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)\geq\varepsilon\right\}\right|=\lim_{r\to\infty}\sum_{j=1}^{n}h_{rj}^{'}\frac{1}{h_{rj}^{'}}\left|\left\{k\in\right.\right.$$

$I'_{rj}: 1-\alpha\left(\frac{x_k}{y_k}-\lambda,\ \mu\right) \geq \varepsilon \ or \ \beta\left(\frac{x_k}{y_k}-\lambda,\ \mu\right) \geq \varepsilon\Big\}\Big| = \lim_{r\to\infty} \frac{1}{h_r}\sum_{j=1}^{n} h'_{rj} t_{rj} = \lim_{r\to\infty}\sum_{j=1}^{n}\frac{h'_{rj}}{h_r} t_{rj}$

Where, $\sum_{j=1}^{n}\frac{h'_{rj}}{h_r} = 1$ and $t_{rj} = \frac{1}{h'_{rj}}\left|\left\{k \in I'_{rj}: 1-\alpha\left(\frac{x_k}{y_k}-\lambda,\ \mu\right) \geq \varepsilon \ or \ \beta\left(\frac{x_k}{y_k}-\lambda,\ \mu\right) \geq \varepsilon\right\}\right|$

Note that; $h'_{rj} \to \infty$ as $r \to \infty, \forall j$

Let. $\lim_{r\to\infty} t_{rj} = \frac{1}{h'_{rj}}\left|\left\{k \in I'_{rj}: 1-\alpha\left(\frac{x_k}{y_k}-\lambda,\ \mu\right) \geq \varepsilon \ or \ \beta\left(\frac{x_k}{y_k}-\lambda,\ \mu\right) \geq \varepsilon\right\}\right| < \varepsilon_j$ where $\varepsilon_j$ is

Sufficiently small for all *j*. Now choose $\varepsilon = max_{1\leq j\leq n}\{\varepsilon_j\}$. Now we have,

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k \in I_r: 1-\alpha\left(\frac{x_k}{y_k}-\lambda,\ \mu\right) \geq \varepsilon \text{ or } \beta\left(\frac{x_k}{y_k}-\lambda,\ \mu\right) \geq \varepsilon\right\}\right|$$

$$= \lim_{r\to\infty}\frac{1}{h_r}\sum_{j=1}^{n} h'_{rj}\varepsilon_j \leq \lim_{r\to\infty}\frac{1}{h_r}\sum_{j=1}^{n} h'_{rj}\varepsilon = \varepsilon \lim_{r\to\infty}\sum_{j=1}^{n}\frac{h'_{rj}}{h_r} = \varepsilon \text{ where } \varepsilon \to 0$$

Thus we obtained,

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k \in I_r: 1-\alpha\left(\frac{x_k}{y_k}-\lambda,\ \mu\right) \geq \varepsilon \ or \ \beta\left(\frac{x_k}{y_k}-\lambda,\ \mu\right) \geq \varepsilon\right\}\right| < \varepsilon$$

Hence theorem is established.

**Theorem 3.7.** Let $\sigma = \{v_r\}$ is a lacunary refinement of lacunary sequence $\theta = \{u_r\}$. Also $[x] = (x_r)$ and $[y] = (y_r)$ be two sequences of non-negative terms of IFNS $(X, \alpha, \beta, *, o)$. If there exist $q > 0$ such that $\frac{h'_r}{h_r} \geq q$ for every $I'_r \subseteq I_r$. Where $I_r = (u_{r-1}, u_r]$ and $I'_r = (v_{r-1}, v_r]$ then $(\alpha,\beta)-\ [x] \overset{S_\theta^\lambda}{\sim} [y] \implies (\alpha,\beta)-\ [x] \overset{S_\sigma^\lambda}{\sim} [y]$

*Proof.* For all $0 < \varepsilon < 1$ and $\mu > 0$. We have

$S_\theta\left\{k \in \mathbb{N}: 1-\alpha\left(\frac{x_k}{y_k}-\lambda,\ \mu\right) \geq \varepsilon \text{ or } \beta\left(\frac{x_k}{y_k}-\lambda,\ \mu\right) \geq \varepsilon\right\} = 0.$ Which implies

$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k \in I_r: 1-\alpha\left(\frac{x_k}{y_k}-\lambda,\ \mu\right) \geq \varepsilon \text{ or } \beta\left(\frac{x_k}{y_k}-\lambda,\ \mu\right) \geq \varepsilon\right\}\right| = 0.$ We now have

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k \in I_r: 1-\alpha\left(\frac{x_k}{y_k}-\lambda,\ \mu\right) \geq \varepsilon \text{ or } \beta\left(\frac{x_k}{y_k}-\lambda,\ \mu\right) \geq \varepsilon\right\}\right| \geq \frac{h'_r}{h_r}\frac{1}{h'_r}\Big|\Big\{k \in I'_r:$$

$$1-\alpha\left(\frac{x_k}{y_k}-\lambda,\ \mu\right) \geq \varepsilon \text{ or } \beta\left(\frac{x_k}{y_k}-\lambda,\ \mu\right) \geq \varepsilon\Big\}\Big| \geq q\frac{1}{h'_r}\Big|\Big\{k \in I'_r: 1-\alpha\left(\frac{x_k}{y_k}-\lambda,\ \mu\right) \geq \varepsilon \text{ or}$$

$$\beta\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)\geq\varepsilon\Big\}\Big|$$ which implies

$$\frac{1}{h_r^{'}}\left|\left\{k\in I_r^{'}:1-\alpha\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)\geq\varepsilon \text{ or } \beta\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)\geq\varepsilon\right\}\right|\leq\frac{1}{q}\frac{1}{h_r}\left|\left\{k\in I_r:1-\alpha\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)\geq\varepsilon \text{ or }\right.\right.$$

$$\left.\left.\beta\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)\geq\varepsilon\right\}\right|$$ Which gives. $$\frac{1}{h_r^{'}}\left|\left\{k\in I_r^{'}:1-\alpha\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)\geq\varepsilon \text{ or } \beta\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)\geq\varepsilon\right\}\right|=0.$$

Or $$S_\sigma\left\{k\in\mathbb{N}:1-\alpha\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)\geq\varepsilon \text{ or } \beta\left(\frac{x_k}{y_k}-\lambda,\ \mu\right)\geq\varepsilon\right\}=0.$$

Thus theorem is established.

**Conclusion :**

Our work extends the ideas of statistical convergence and lacunary statistically convergence to asymptotic statistical convergence, asymptotic lacunary statistically convergence and strongly asymptotic lacunary statistically convergence in Intuitionistic fuzzy normed space which are more general and geometrically sound in study. Our results are also analogus of results discussed in [6, 7] by R. Patterson & E. Savas.

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