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Solutions of Fractional Differential Equations by Variation of Parameters

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Abstract - The current research work looks at how to use parameter variation to obtain a specific result to a non-homogeneous linear fractional differential equation. It is feasible to develop a formula for ordinary differential equations that is comparable to that of ordinary differential equations.

Index Terms - Solutions, fractional differential equation, variation of parameters

INTRODUCTION

The fractional derivatives have many definitions available in the literature. The reference of the Fractional calculus can be found in the end of 17th century. The ½ order derivative was discussed by Leibniz and L'Hospital in 1695. This period is also regarded as the "birth of the fractional differential equation".. The fractional calculus did not gather enough attention for a long period of time. Only few mathematicians and scientists considered and studied this, but over the last few decades, the fractional calculus has so far been studied extensively in field of engineering and applied sciences. The utilization and operation of fractional derivative and the function of fractional calculus in mathematical sciences and engineering has thus been ascertained by various authors [1]. [2], [3], [4] and [6]. The R L and the Caputo definitions are the most important, and for other utilization, see [5, 7, 8, 9,10]. Riemann-Liouville Definition: For $\alpha \in (n-1, n)$ the α derivative of *f* is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \frac{d^n}{dt^n} \int_{\alpha}^{t} \frac{f(x)}{(t-x)^{\alpha-n+1}}\, dx$$

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \int_{\alpha}^{t} \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}}\, dx$$

There are numerous drawbacks to such definitions, including
The established formula is not satisfied by all fractional derivatives: $D_a^{\alpha}(fg) = f\,D_a^{\alpha}(g) + gD_a^{\alpha}(f)$

i. The fractional derivatives does not fully satisfies the widely accepted function for the derivative of the quotient of the two functions.

$$D_a^{\alpha}(f/g) = \frac{gD_a^{\alpha}(f) - f\,D_a^{\alpha}(g)}{g^2}$$

ii. The chain rule is not satisfied by all functional derivatives:

$$D_a^{\alpha}(f \;o\; g)(t) = f^{(\alpha)}(g(t))g^{(\alpha)}(t)$$

iii. In general, $D^{\alpha} D^{\beta} f = D^{\alpha+\beta} f$, is not appeased by all the fractional derivatives.
iv. It is assumed that the function *f* is differentiable by all fractional derivatives, and especially Caputo dentition.

We have the following properties according to this definition:

1 $T_\alpha(1) = 0$,
2 $T_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in$ R,
3 $T_\alpha(\sin at) = at^{1-\alpha} \cos at,\ a \in$ R, 4
4 $T_\alpha(\cos at) = -at^{1-\alpha} \sin at,\ a \in$ R,
5 $T_\alpha(e^{at}) = at^{1-\alpha}e^{at},\ a \in$ R.

Numerous functions also respond as they would in a conventional derivative. Few examples of the formula are mentioned here:

$$T_\alpha\left(\frac{1}{\alpha}t^{\alpha}\right) = 1$$
$$T_\alpha\left(e^{\frac{1}{\alpha}t^{\alpha}}\right) = e^{\frac{1}{\alpha}t^{\alpha}}$$
$$T_\alpha\left(sin\frac{1}{\alpha}t^{\alpha}\right) = cos\left(\frac{1}{\alpha}t^{\alpha}\right)$$
$$T_\alpha\left(cos\frac{1}{\alpha}t^{\alpha}\right) = -sin\left(\frac{1}{\alpha}t^{\alpha}\right)$$

The conformable fractional derivative is used in research to investigate strategies to discover specific result to a segment of non-homogenous linear fractional differential equations.

Definition 2.1.

$$T^{n\alpha} f = D^{\alpha} D^{\alpha} \dots . D^{n} f$$

For simplicity, we write $f^{(n\alpha)}\ for\ T^{n\alpha}.\ So\ y^{(2\alpha)}(x)\ \ stands\ for\ \frac{d^{\alpha}}{dx^{\alpha}}\left(\frac{d^{\alpha}y}{dx^{\alpha}}\right)$

Definition 2.2.

$T^{n\alpha}y + a_{n-1}T^{(n-1)\alpha}y + \dots + a_1T^{\alpha}y + a_0y = f(x)$ (2.1)

is known as a linear fractional differential equation of order n.

The coefficients $a_0, a_1, \dots, a_{n-1}$ could be constants or variables.

Since $0 < \alpha < 1$, if y is n times differentiable, then there are n-independent solutions $y_1, y_2, \dots, y_n$ for the homogeneous differential equation

$T^{n\alpha} + \dots + a_0y = 0$ (2.2)

The indeterminate coefficients or the variation of parameters method can be brought into use to find a specific solution for the equation 2.1

When $n = 2$, we can discover a complete solution for y_p

Definition 2.3. Let's consider independent functions y_1, y_2

$$W^{\alpha}[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1^{(\alpha)} & y_2^{(\alpha)} \end{vmatrix}$$

is defined as the α−Wronskian of y_1 and y_2.

Generally speaking, if $y_1, y_2, \dots, y_n$ are n linearly independent functions,

$$W^{\alpha}[y_1, y_2, \dots, y_n] = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1^{(\alpha)} & y_2^{(\alpha)} & \dots & y_n^{(\alpha)} \\ . & & & \\ . & & & \\ . & & & \\ y_1^{((n-1)\alpha)} & y_2^{((n-1)\alpha)} & \dots & y_n^{((n-1)\alpha)} \end{vmatrix}$$

THE MAIN RESULTS

We consider two independent solutions y_1 and y_2 for the equation

$D^{\alpha}(D^{\alpha}y) + a_1D^{\alpha}y + a_0y = 0$ (3.1)

The goal is to find y_p for

Method:

$D^{\alpha}(D^{\alpha}y) + a_1D^{\alpha}y + a_0y = f(x)$ (3.2)

Let

$y_p = c_1y_1 + c_2y_2$

where c_1 and c_2 are functions of x. So,

$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$

$y_p^{(\alpha)} = c_1^{(\alpha)}y_1 + c_1y_1^{(\alpha)} + c_2^{(\alpha)}y_2 + c_2y_2^{(\alpha)}$

Put

$c_1^{(\alpha)}y_1 + c_2^{(\alpha)}y_2 = 0$ (3.3)

Hence,

$D^{\alpha}y_p^{(\alpha)} = D^{\alpha}(c_1y_1 + c_2y_2) = c_1^{(\alpha)}y_1^{(\alpha)} + c_1y_1^{(2\alpha)} + c_2^{(\alpha)}y_2^{(\alpha)} + c_2y_2^{(2\alpha)}$

Since y_1 and y_2 are solutions for 3.2, we get:

$c_1^{(\alpha)}y_1^{(\alpha)} + c_2^{(\alpha)}y_2^{(\alpha)} = f(x)$ (3.4)

Solving 3.3 and 3.4 we get

$$y_p(x) = \int_a^x \frac{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1(x) & y_2(x) \end{vmatrix}}{W^{\alpha}[y_1, y_2](t)} \frac{f(t)}{t^{1-\alpha}}\, dt\,,$$

that is to say

$$y_p(x) = I_\alpha^a \left[\frac{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1(x) & y_2(x) \end{vmatrix}}{W^\alpha[y_1, y_2](t)} f(t) \right]$$
$$= I_\alpha^a\big(K(x,t)f(t)\big),$$
where;
$$K(x,t) = \frac{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1(x) & y_2(x) \end{vmatrix}}{W^\alpha[y_1, y_2](t)}$$

A specific solution of the equation (3.1) can be found as
$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x) + ... + c_n(x)y_n(x)$. (3.5)

Example 3.1. Let us first try to provide a solution to the following
$$D^{1/2}\, D^{1/2}\, y = \frac{x}{(x^2+1)}$$
We can find $y_1 = 1,\ \ y_2 = \sqrt{x}$ are two solutions for above equation. Now let us solve the particular solutions of equation 1 using the formula
$$y_{p1} = -y_1 \int \frac{y_2(x)\, f(x)}{W(y_1\, y_2)x^{2-2\propto}}\, dx$$
$$y_{p2} = -y_2 \int \frac{y_1(x)\, f(x)}{W(y_1\, y_2)x^{2-2\propto}}\, dx$$
we have, $y_1 = 1,\ \ y_2 = \sqrt{x}$, $f(x) = \frac{x}{x^2+1}$
$$W(1, \sqrt{x}) = \begin{vmatrix} 1 & \sqrt{x} \\ 0 & 1/2\sqrt{x} \end{vmatrix} = 1/2\sqrt{x}$$
$$y_{p1}(x) = -2 \int \frac{1}{(x^2+1)} dx$$
$$= -2\tan^{-1} x$$
$$y_{p2}(x) = 2\sqrt{x} \int \frac{1}{(x^2+1)} \frac{1}{\sqrt{x}} dx$$
$$= 2\sqrt{x}\, arc \tan x + \mathrm{C}$$
Hence, we can get general solution of above equation y= $y_{p1}(x) + y_{p2}(x) + 1 + \sqrt{x}$

Example 3.2. We will solve another fractional differential equation
$$XD^{1/2}\left(D^{\frac{1}{2}}y\right) - \frac{y}{2} = x^3$$
We introduce two possible solutions for the above-mentioned equation:
$$y_1 = x,\ \ and\ \ y_2 = x^{-1/2}$$
$$y_{p1} = -x \int \frac{x^{-1/2}\, x^2}{-\frac{3}{2}\, x^{-1/2} x}\, dx$$
$$y_{p1} = +x \int \frac{2}{3} x\, dx$$
$$= \frac{2}{9}\, x^3$$
$$y_{p2} = y_2 \int \frac{y_1(x)\, f(x)}{W(y_1\, y_2)x^{2-2\propto}}\, dx$$
$$-\frac{2}{3} x^{-1/2} \int x^{5/2}\, dx$$
$$= \frac{-4}{21} x^{3/2}$$
Hence, general solution will be
$$y = c_1 x + c_2 x^{-1/2} + \frac{2}{9}x^3 - \frac{4}{21}x^3$$

Example 3.3. Let us solve another fractional differential equation
$$XD^{1/2}\left(D^{\frac{1}{2}}y\right) - \frac{y}{2} = 8x^3\, e^x$$
Let
$y_1 = x,\ \ y_2 = x^{-1/2}$ two possible explanations for the above equation

$$y_{p1} = -y_1 \int \frac{y_2(x)\, f(x)}{W(y_1\, y_2) x^{2-2\propto}}\, dx$$
$$= 16x \int e^x dx$$
$$= 16x\, e^x$$
$$y_{p2} = x^{-1/2} \int \frac{x\, 8x^2 e^x}{-\frac{1}{2} x^{1/2} x}\, dx$$
$$y_{p2} = -2x^{-1/2} \int x^{3/2} e^x\, dx$$
$$= -2xe^x \sum_{s=0}^{m} (-1)^m (1)^{-s+1} \frac{\left(\frac{3}{2}\right)!}{\left(\frac{3}{2} - s\right)!} x^{-s}$$

Hence, we can get general solution of above equation by adding y_1, y_2, y_{p1} *and* y_{p2}.

Conclusion: We have got the solution very easily and efficiently using variation of parameter for fractional differential equation. We found that variation of parameter is quite helpful in obtaining a specific solution to a non-homogeneous linear fractional differential equation. Various fractional differential equations can be solved after applying this method.

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