Fuzzy Bipolar Pythagorean Graphs with Laplacian Energy

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Abstract - In this paper, we process the energy and fuzzy Bipolar Pythagorean graphs with Laplacian energy (FBPGs). Besides, we determine the lower and upper limits for the energy and Laplacian energy of FBPGs.

Index Terms - Energy(E), Pythagorean fuzzy graphs (PFGs); Laplacian energy(LE), Pythagorean fuzzy set(PFS).

INTRODUCTION

Yager as of late [19,20] presented the idea of the PFS as a speculation of intuitionistic fuzzy set(IFS) [3] to deal with perplexing inaccuracy and vulnerability in sensible decision-making issues. The beginning of PFS by Yager [20]. The idea of Pythagorean fuzzy number (PFN) and numerical type of PFS was presented by Xu and Zhang [21]. In the meantime, they introduced a series of fundamental of PFNs and projected Pythagorean fluffy aggregate operators. PFS, a clever class of the non-standard fuzzy set, has a wide-ranging scope of uses in various area, for example, clinical analysis [13], Internet corporate security [14], the assistance nature of homegrown aircrafts [21] also, the lead representative choice of the Bank [17].

Graph portrayals are the most part utilized for managing primary data, in various areas, for example, activities research, organizations, frameworks examination, design acknowledgment, financial aspects and picture translation. In graph the idea of E was presented by Gutman [4] due to its pertinence of specific atoms and discovered lower and upper limits for the energy of graphs [5]. Afterward, Gutman and Zhou [6] characterized the Laplacian energy of a graph(LEG). The idea of fuzzy graph [7], in view of fuzzy relations [22]. Rosenfeld [18] examined the idea of the fuzzy graph and fostered its construction. The energy of a fuzzy graph was researched in [2] by Mathew and Anjali. The LEFG was characterized by Fayazi and Sharbaf [16]. Karunambigai and Parvathi [12] summed up the idea of intuitionistic fuzzy graph (IFG). Afterward, strong IFGs is argued by Davvaz and Akram [1]. Praba [15] characterized the energy of IFGs as an expansion of [2]. Rajeshwari, Murugesan and Venkatesh [8] summed up the idea of LEBFG's. As of late, Naz et al. [11] suggested the idea of PFGs, a speculation of the idea of Davvaz and Akram IFGs [1], alongside its applications in decision making. The ideas of E and LEPFGs was explored by Muhammad Akram and Suera Naz [10].

Energy of FBPG

Definition 2.1. Let G= (S, D) be the FBPG on a non-empty set Q, then S is the FBPG on Q and D is the BPF relation on Q such that

$$\mu^P_{D_W}(xy) \le \min\{\mu^P_{S_W}(x), \mu^P_{S_W}(y)\}$$
$$\mu^P_{D_U}(xy) \ge \max\{\mu^P_{S_U}(x), \mu^P_{S_U}(y)\}$$
$$\mu^N_{D_W}(xy) \ge \max\{\mu^N_{S_W}(x), \mu^N_{S_W}(y)\}$$
$$\mu^N_{D_U}(xy) \le \min\{\mu^N_{S_u}(x), \mu^N_{S_u}(y)\}$$

Definition 2.2. Let G = (S, D) be the FBPG then the adjacency matrix $A(G) = [a_{ij}]$ where $a_{ij} = \left(\mu^P_{D_W}(q_i q_j), \mu^P_{D_U}(q_i q_j), \mu^N_{D_W}(q_i q_j), \mu^N_{D_U}(q_i q_j)\right)$, where $\mu^P_{D_W}(q_i q_j)$ and $\mu^N_{D_W}(q_i q_j)$ denotes the positive and negative strength between q_i and q_j, $\mu^P_{D_U}(q_i q_j)$ and $\mu^N_{D_U}(q_i q_j)$ denotes the positive and negative non-membership strength between q_i and q_j.

Definition 2.3. In FBPG, the spectrum of connection matrix is indicated by the arrangement of eigenvalues.

$A(G) = \left(A(\mu^P_{D_W}(q_i q_j)), A(\mu^P_{D_U}(q_i q_j)), A(\mu^N_{D_W}(q_i q_j)), A(\mu^N_{D_U}(q_i q_j))\right)$ and energy is denoted by

$$E(G) = \left(E(\mu^P_{D_W}(q_i q_j)), E(\mu^P_{D_U}(q_i q_j)), E(\mu^N_{D_W}(q_i q_j)), E(\mu^N_{D_U}(q_i q_j))\right) = \left(\sum_{i=1}^{n}|\lambda_i|, \sum_{i=1}^{n}|\alpha_i|, \sum_{i=1}^{n}|\zeta_i|, \sum_{i=1}^{n}|\beta_i|\right).$$

Example 2.1.

For FBPG the adjacency matrix is given by

$$A(G)=\begin{bmatrix} (0,0,0,0) & (0.1,0.8,-0.1,-0.5) & (0,0,0,0) & (0.4,0.7,-0.5,-0.5) & (0,0,0,0) \\ (0.1,0.8,-0.1,-0.5) & (0,0,0,0) & (0.2,0.8,-0.2,-0.4) & (0,0,0,0) & (0,0,0,0) \\ (0,0,0,0) & (0.2,0.8,-0.2,-0.4) & (0,0,0,0) & (0.3,0.6,-0.4,-0.5) & (0.5,0.4,-0.2,-0.4) \\ (0.4,0.7,-0.5,-0.5) & (0,0,0,0) & (0.3,0.6,-0.4,-0.5) & (0,0,0,0) & (0.5,0.5,-0.4,-0.5) \\ (0,0,0,0) & (0,0,0,0) & (0.5,0.4,-0.2,-0.4) & (0.5,0.5,-0.4,-0.5) & (0,0,0,0) \end{bmatrix}$$

The spectrum and the energy of FBPG,
Spec(G) = {(0.9472, 1.5796, 0.8406, -1.1613), (-0.6531, -1.4591, 0.7264, 0.9605), (-0.5, -0.5545, 0.2702, 0.4957), (0.1531, 0.3815, -0.0942, -0.3470), (0.0528, 0.0525, -0.0618, 0.0521)}
E(G) = (2.3062, 4.0272, 1.9932, 3.0166)

Theorem 2.1. Let G be the undirected FBPG, then the eigenvalue of the connection matrix is indicated by $A(\mu^P_{D_W}(q_iq_j)), A(\mu^P_{D_U}(q_iq_j)), A(\mu^N_{D_W}(q_iq_j))$ and $A(\mu^N_{D_U}(q_iq_j))$ respectively then:

$\sum_{i=1}^{n}|\lambda_i|=0, \sum_{i=1}^{n}|\alpha_i|=0, \sum_{i=1}^{n}|\zeta_i|=0$ and $\sum_{i=1}^{n}|\beta_i|=0$

$\sum_{i=1}^{n}\lambda_i^2 = 2\sum_{1\le i<j\le n}(\mu^P_{D_W}(q_iq_j))^2, \sum_{i=1}^{n}\alpha_i^2 = 2\sum_{1\le i<j\le n}(\mu^P_{D_U}(q_iq_j))^2, \sum_{i=1}^{n}\zeta_i^n = 2\sum_{1\le i<j\le n}(\mu^N_{D_W}(q_iq_j))^2$ and

$\sum_{i=1}^{n}\beta_i^2 = 2\sum_{1\le i<j\le n}(\mu^N_{D_U}(q_iq_j))^2$

Proof. (i) Let G be the FBPG then A(G) is a square matrix that is equivalent to its transpose and sum of diagonal is zero, then

$$\sum_{i=1}^{n}|\lambda_i|=0, \sum_{i=1}^{n}|\alpha_i|=0, \sum_{i=1}^{n}|\zeta_i|=0 \text{ and } \sum_{i=1}^{n}|\beta_i|=0$$

We can deduce the following from the properties sum of diagonal matrix:

$$tr\left(A(\mu^P_{D_W}(q_iq_j))^2\right)=\sum_{i=1}^{n}\lambda_i^2$$

Where

$$\begin{aligned} tr\left(A(\mu^P_{D_W}(q_iq_j))^2\right) &= \left(0+(\mu^P_{D_W}(q_1q_2))^2+\ldots+(\mu^P_{D_W}(q_1q_n))^2\right) \\ &\quad + \left((\mu^P_{D_W}(q_2q_1))^2+0+\ldots+(\mu^P_{D_W}(q_1q_n))^2\right) \\ &\quad + \ldots + \left((\mu^P_{D_W}(q_nq_1))^2+(\mu^P_{D_W}(q_2q_n))^2+\ldots+0\right) \\ &= 2\sum_{1\le i<j\le n}(\mu^P_{D_W}(q_iq_j))^2 \end{aligned}$$

$$\therefore \sum_{i=1}^{n}\lambda_i^2 = 2\sum_{1\le i<j\le n}(\mu^P_{D_W}(q_iq_j))^2$$

Similarly

$$\sum_{i=1}^{n}\alpha_i^2 = 2\sum_{1\le i<j\le n}(\mu^P_{D_U}(q_iq_j))^2, \sum_{i=1}^{n}\zeta_i^n = 2\sum_{1\le i<j\le n}(\mu^N_{D_W}(q_iq_j))^2 \text{ and } \sum_{i=1}^{n}\beta_i^2 = 2\sum_{1\le i<j\le n}(\mu^N_{D_U}(q_iq_j))^2$$

Theorem 2.2. Let G be the undirected FBPG with m vertices and the connection matrix

$$A(G)=\left(A(\mu^P_{D_W}(q_iq_j)), A(\mu^P_{D_U}(q_iq_j)), A(\mu^N_{D_W}(q_iq_j)), A(\mu^N_{D_U}(q_iq_j))\right) \text{ then}$$

$$(i)\sqrt{2\sum_{1\le i<j\le m}(\mu^P_{D_W}(q_iq_j))^2+m(m-1)\left|\det\left(A(\mu^P_{D_W}(q_iq_j))^2\right)\right|^{\frac{2}{m}}} \le E(\mu^P_{D_W}(q_iq_j)) \le \sqrt{2m\sum_{1\le i<j\le m}(\mu^P_{D_W}(q_iq_j))^2}$$

$$(ii)\sqrt{2\sum_{1\le i<j\le m}\left(\mu^{P}_{D_U}(q_iq_j)\right)^2+m(m-1)\left|\det\left(A\left(\mu^{P}_{D_U}(q_iq_j)\right)^2\right)\right|^{\frac{2}{m}}}\le E\left(\mu^{P}_{D_U}(q_iq_j)\right)\le\sqrt{2m\sum_{1\le i<j\le m}\left(\mu^{P}_{D_U}(q_iq_j)\right)^2}$$

$$(iii)\sqrt{2\sum_{1\le i<j\le m}\left(\mu^{N}_{D_W}(q_iq_j)\right)^2+m(m-1)\left|\det\left(A\left(\mu^{N}_{D_W}(q_iq_j)\right)^2\right)\right|^{\frac{2}{m}}}\le E\left(\mu^{N}_{D_W}(q_iq_j)\right)\le\sqrt{2m\sum_{1\le i<j\le m}\left(\mu^{N}_{D_W}(q_iq_j)\right)^2}$$

$$(iv)\sqrt{2\sum_{1\le i<j\le m}\left(\mu^{N}_{D_U}(q_iq_j)\right)^2+m(m-1)\left|\det\left(A\left(\mu^{N}_{D_U}(q_iq_j)\right)^2\right)\right|^{\frac{2}{m}}}\le E\left(\mu^{N}_{D_U}(q_iq_j)\right)\le\sqrt{2m\sum_{1\le i<j\le m}\left(\mu^{N}_{D_U}(q_iq_j)\right)^2}$$

Proof.

(i)Upper Bound:

With m entries, use the Cauchy-schwarz inequality.

$$\sum_{i=1}^{m}|\lambda_i|=\sqrt{m}\sqrt{\sum_{i=1}^{m}|\lambda_i|^2}$$

$$\left(\sum_{i=1}^{m}\lambda_i\right)^2=\sum_{i=1}^{m}|\lambda_i|^2+2\sum_{1\le i<j\le m}\lambda_i\lambda_j$$

In characteristic polynomial if we compare the coefficients of λ^{n-2}

$$\prod_{i=1}^{m}(\lambda-\lambda_i)=|A(G)-\lambda I|$$

$$\sum_{1\le i<j\le m}\lambda_i\lambda_j=-\sum_{1\le i<j\le m}\left(\mu^{P}_{D_W}(q_iq_j)\right)^2$$

Replacing (3) in (2)

$$\sum_{i=1}^{m}|\lambda_i|^2=2\sum_{1\le i<j\le m}\left(\mu^{P}_{D_W}(q_iq_j)\right)^2$$

Replacing (4) in (1)

$$\sum_{i=1}^{m}|\lambda_i|=\sqrt{2n\sum_{1\le i<j\le m}\left(\mu^{P}_{D_W}(q_iq_j)\right)^2}$$

Therefore,

$$E\left(\mu^{P}_{D_W}(q_iq_j)\right)\le\sqrt{2m\sum_{1\le i<j\le m}\left(\mu^{P}_{D_W}(q_iq_j)\right)^2}$$

Lower bound:

$$\left(E(\mu^{P}_{D_W}(q_iq_j)\right)^2=\left(\sum_{i=1}^{m}|\lambda_i|\right)^2$$

$$=\sum_{i=1}^{m}|\lambda_i|^2+2\sum_{1\le i<j\le m}\lambda_i\lambda_j$$

$$=2\sum_{1\le i<j\le n}\left(\mu^{P}_{D_W}(q_iq_j)\right)^2+m(m-1)AM\left\{|\lambda_i\lambda_j|\right\}$$

$$AM\left\{|\lambda_i\lambda_j|\right\}\ge GM\left\{|\lambda_i\lambda_j|\right\},1\le i<j\le m$$

$$E\left(\mu_{D_W}^P\left(q_i q_j\right)\right) \geq \sqrt{2\sum_{1\leq i<j\leq m}\left(\mu_{D_W}^P\left(q_i q_j\right)\right)^2 + m(m-1)GM\left\{\left|\lambda_i\lambda_j\right|\right\}}$$

$$GM\left\{\left|\lambda_i\lambda_j\right|\right\} = \left(\prod_{1\leq i<j\leq m}\left|\lambda_i\lambda_j\right|\right)^{\frac{2}{m(m-1)}}$$

$$= \left(\prod_{i=1}^{m}\left|\lambda_i\right|^{m-1}\right)^{\frac{2}{m(m-1)}}$$

$$= \left(\prod_{i=1}^{m}\left|\lambda_i\right|\right)^{\frac{2}{m}}$$

$$= \left|\det\left(A\left(\mu_{D_W}^P\left(q_i q_j\right)\right)\right)\right|^{\frac{2}{m}}$$

$$E\left(\mu_{ij}^P(G)\right) \geq \sqrt{2\sum_{1\leq i<j\leq m}\left(\mu_{D_W}^P\left(q_i q_j\right)\right)^2 + m(m-1)\left|\det\left(A\left(\mu_{D_W}^P\left(q_i q_j\right)\right)\right)\right|^{\frac{2}{m}}}$$

Thus $\sqrt{2\sum_{1\leq i<j\leq m}\left(\mu_{D_W}^P\left(q_i q_j\right)\right)^2 + m(m-1)\left|\det\left(A\left(\mu_{D_W}^P\left(q_i q_j\right)\right)^2\right)\right|^{\frac{2}{m}}} \leq E\left(\mu_{D_W}^P\left(q_i q_j\right)\right) \leq \sqrt{2m\sum_{1\leq i<j\leq m}\left(\mu_{D_W}^P\left(q_i q_j\right)\right)^2}$

Likewise, we can demonstrate

$$\sqrt{2\sum_{1\leq i<j\leq m}\left(\mu_{D_U}^P\left(q_i q_j\right)\right)^2 + m(m-1)\left|\det\left(A\left(\mu_{D_U}^P\left(q_i q_j\right)\right)^2\right)\right|^{\frac{2}{m}}} \leq E\left(\mu_{D_U}^P\left(q_i q_j\right)\right) \leq \sqrt{2m\sum_{1\leq i<j\leq m}\left(\mu_{D_U}^P\left(q_i q_j\right)\right)^2}$$

$$\sqrt{2\sum_{1\leq i<j\leq m}\left(\mu_{D_W}^N\left(q_i q_j\right)\right)^2 + m(m-1)\left|\det\left(A\left(\mu_{D_W}^N\left(q_i q_j\right)\right)^2\right)\right|^{\frac{2}{m}}} \leq E\left(\mu_{D_W}^N\left(q_i q_j\right)\right) \leq \sqrt{2m\sum_{1\leq i<j\leq m}\left(\mu_{D_W}^N\left(q_i q_j\right)\right)^2}$$

$$\sqrt{2\sum_{1\leq i<j\leq m}\left(\mu_{D_U}^N\left(q_i q_j\right)\right)^2 + m(m-1)\left|\det\left(A\left(\mu_{D_U}^N\left(q_i q_j\right)\right)^2\right)\right|^{\frac{2}{m}}} \leq E\left(\mu_{D_U}^N\left(q_i q_j\right)\right) \leq \sqrt{2m\sum_{1\leq i<j\leq m}\left(\mu_{D_U}^N\left(q_i q_j\right)\right)^2}$$

Theorem 2.3. Let G be the undirected FBPG with m vertices and the connection matrix $A(G) = \left(A\left(\mu_{D_W}^P\left(q_i q_j\right)\right), A\left(\mu_{D_U}^P\left(q_i q_j\right)\right), A\left(\mu_{D_W}^N\left(q_i q_j\right)\right), A\left(\mu_{D_U}^N\left(q_i q_j\right)\right)\right)$. If $m \leq 2\sum_{1\leq i<j\leq m}\left(\mu_{D_W}^P\left(q_i q_j\right)\right)^2, m \leq 2\sum_{1\leq i<j\leq m}\left(\mu_{D_U}^P\left(q_i q_j\right)\right)^2, m \leq 2\sum_{1\leq i<j\leq m}\left(\mu_{D_W}^N\left(q_i q_j\right)\right)^2$ and $m \leq 2\sum_{1\leq i<j\leq m}\left(\mu_{D_U}^N\left(q_i q_j\right)\right)^2$, then

$$(i)\, E\left(\mu_{D_W}^P\left(q_i q_j\right)\right) \leq \frac{2\sum_{1\leq i<j\leq m}\left(\mu_{D_W}^P\left(q_i q_j\right)\right)^2}{m} + \sqrt{(m-1)\left\{2\sum_{1\leq i<j\leq m}\left(\mu_{D_W}^P\left(q_i q_j\right)\right)^2 - \left(\frac{2\sum_{1\leq i<j\leq m}\left(\mu_{D_W}^P\left(q_i q_j\right)\right)^2}{m}\right)^2\right\}}$$

$$(ii)\, E\left(\mu_{D_U}^P\left(q_i q_j\right)\right) \leq \frac{2\sum_{1\leq i<j\leq m}\left(\mu_{D_U}^P\left(q_i q_j\right)\right)^2}{m} + \sqrt{(m-1)\left\{2\sum_{1\leq i<j\leq m}\left(\mu_{D_U}^P\left(q_i q_j\right)\right)^2 - \left(\frac{2\sum_{1\leq i<j\leq m}\left(\mu_{D_U}^P\left(q_i q_j\right)\right)^2}{m}\right)^2\right\}}$$

$$(iii)\, E\left(\mu_{D_W}^{N}\left(q_iq_j\right)\right)\le \frac{2\sum_{1\le i<j\le m}\left(\mu_{D_W}^{N}\left(q_iq_j\right)\right)^2}{m}+\sqrt{(m-1)\left\{2\sum_{1\le i<j\le m}\left(\mu_{D_W}^{N}\left(q_iq_j\right)\right)^2-\left(\frac{2\sum_{1\le i<j\le m}\left(\mu_{D_W}^{N}\left(q_iq_j\right)\right)^2}{m}\right)^2\right\}}$$

$$(iv)\, E\left(\mu_{D_U}^{N}\left(q_iq_j\right)\right)\le \frac{2\sum_{1\le i<j\le m}\left(\mu_{D_U}^{N}\left(q_iq_j\right)\right)^2}{m}+\sqrt{(m-1)\left\{2\sum_{1\le i<j\le m}\left(\mu_{D_U}^{N}\left(q_iq_j\right)\right)^2-\left(\frac{2\sum_{1\le i<j\le m}\left(\mu_{D_U}^{N}\left(q_iq_j\right)\right)^2}{m}\right)^2\right\}}$$

Laplacian Energy of Bipolar Pythagorean Fuzzy Graphs

Definition 3.1. In the FBPG the Laplacian matrix is defined as the difference between the degree and adjacency matrix, denoted as L(BG)=D(BG)-A(BG) Laplacian matrix of FBPG can be written as $L\left(\mu_{D_W}^{P}\left(q_iq_j\right)\right), L\left(\mu_{D_U}^{P}\left(q_iq_j\right)\right), L\left(\mu_{D_W}^{N}\left(q_iq_j\right)\right)$ and $L\left(\mu_{D_U}^{N}\left(q_iq_j\right)\right)$

$$L(BG)=\left[L\left(\mu_{D_W}^{P}\left(q_iq_j\right)\right), L\left(\mu_{D_U}^{P}\left(q_iq_j\right)\right), L\left(\mu_{D_W}^{N}\left(q_iq_j\right)\right), L\left(\mu_{D_U}^{N}\left(q_iq_j\right)\right)\right]$$

Definition 3.2. In FBPG, the spectrum of Laplacian matrix is denoted by the set of Laplacian eigenvalues. For FBPG the Laplacian energy is denoted by

$$L(BG)=\left(\sum_{i=1}^{m}|\alpha_i|, \sum_{i=1}^{m}|\beta_i|, \sum_{i=1}^{m}|\chi_i|, \sum_{i=1}^{m}|\rho_i|\right)$$

Where

$$\alpha_i=\upsilon_i-\frac{2\sum_{1\le i<j\le m}\left(\mu_{D_W}^{P}\left(q_iq_j\right)\right)}{m},\quad \beta_i=\zeta_i-\frac{2\sum_{1\le i<j\le m}\left(\mu_{D_U}^{P}\left(q_iq_j\right)\right)}{m},\quad \chi_i=\tau_i-\frac{2\sum_{1\le i<j\le m}\left(\mu_{D_W}^{N}\left(q_iq_j\right)\right)}{m} \text{ and}$$

$$\rho_i=\xi_i-\frac{2\sum_{1\le i<j\le m}\left(\mu_{D_U}^{N}\left(q_iq_j\right)\right)}{m}$$

Theorem 3.1. Let G be the FBPG, then the Laplacian matrix of G is denoted by L(BG). If $\upsilon_i\ge\upsilon_i+1$, $\zeta_i\ge\zeta_{i+1}$, $\tau_i\ge\tau_{i+1}$ and $\xi_i\ge\xi_{i+1}$ where i=1,2,…,m-1 are the eigenvalues of $L\left(\mu_{D_W}^{P}\left(q_iq_j\right)\right), L\left(\mu_{D_U}^{P}\left(q_iq_j\right)\right), L\left(\mu_{D_W}^{N}\left(q_iq_j\right)\right)$ and $L\left(\mu_{D_U}^{N}\left(q_iq_j\right)\right)$ respectively,

$$\sum_{i=1}^{m}\upsilon_i=2\sum_{1\le i<j\le m}\mu_{D_W}^{P}\left(q_iq_j\right),\ \sum_{i=1}^{m}\zeta_i=2\sum_{1\le i<j\le m}\mu_{D_U}^{P}\left(q_iq_j\right),\ \sum_{i=1}^{m}\tau_i=2\sum_{1\le i<j\le m}\mu_{D_W}^{N}\left(q_iq_j\right) \text{ and } \sum_{i=1}^{m}\xi_i=2\sum_{1\le i<j\le m}\mu_{D_U}^{N}\left(q_iq_j\right).$$

$$\sum_{i=1}^{m}\upsilon_i^2=2\sum_{1\le i<j\le m}\left(\mu_{D_W}^{P}\left(q_iq_j\right)\right)^2+\sum_{i=1}^{m}d^2_{\mu_{D_W}^{P}\left(q_iq_j\right)}\left(q_i\right),\ \sum_{i=1}^{m}\zeta_i^2=2\sum_{1\le i<j\le m}\left(\mu_{D_U}^{P}\left(q_iq_j\right)\right)^2+\sum_{i=1}^{m}d^2_{\mu_{D_U}^{P}\left(q_iq_j\right)}\left(q_i\right),$$

$$\sum_{i=1}^{m}\tau_i^2=2\sum_{1\le i<j\le m}\left(\mu_{D_W}^{N}\left(q_iq_j\right)\right)^2+\sum_{i=1}^{m}d^2_{\mu_{D_W}^{N}\left(q_iq_j\right)}\left(q_i\right),$$

$$\sum_{i=1}^{m}\xi_i^2=2\sum_{1\le i<j\le m}\left(\mu_{D_U}^{N}\left(q_iq_j\right)\right)^2+\sum_{i=1}^{m}d^2_{\mu_{D_U}^{N}\left(q_iq_j\right)}\left(q_i\right)$$

Proof. If L(G) be the Laplacian matrix on G, then

$$\sum_{i=1}^{m}\upsilon_i=tr\left(L(G)\right)=\sum_{i=1}^{m}d^2_{\mu_{D_W}^{P}\left(q_iq_j\right)}\left(q_i\right)=2\sum_{1\le i<j\le m}\mu_{D_W}^{P}\left(q_iq_j\right).$$

Therefore,

$$\sum_{i=1}^{m}\zeta_i=2\sum_{1\le i<j\le m}\mu_{D_U}^{P}\left(q_iq_j\right),\ \sum_{i=1}^{m}\tau_i=2\sum_{1\le i<j\le m}\mu_{D_W}^{N}\left(q_iq_j\right) \text{ and } \sum_{i=1}^{m}\xi_i=2\sum_{1\le i<j\le m}\mu_{D_U}^{N}\left(q_iq_j\right).$$

By the definition of Laplacian matrix and by trace properties of matrix, we have

$$tr\left(L\left(\mu^{P}_{D_W}\left(q_i q_j\right)\right)^2\right)=\sum_{i=1}^{m}\upsilon_i^2$$

$$\begin{aligned}tr\left(L\left(\mu^{P}_{D_W}\left(q_i q_j\right)\right)^2\right) &= \left(d^{2}_{\mu^{P}_{D_W}\left(q_i q_j\right)}\left(q_1\right)+\left(\mu^{P}_{D_W}\left(q_1 q_2\right)\right)^2+\ldots+\left(\mu^{P}_{D_W}\left(q_1 q_m\right)\right)^2\right) \\ &+\left(\left(\mu^{P}_{D_W}\left(q_2 q_1\right)\right)^2+d^{2}_{\mu^{P}_{D_W}\left(q_i q_j\right)}\left(q_2\right)+\ldots+\left(\mu^{P}_{D_W}\left(q_1 q_m\right)\right)^2\right) \\ &+\ldots+\left(\left(\mu^{P}_{D_W}\left(q_m q_1\right)\right)^2+\left(\mu^{P}_{D_W}\left(q_2 q_m\right)\right)^2+\ldots+d^{2}_{\mu^{P}_{D_W}\left(q_i q_j\right)}\left(q_m\right)\right) \\ &= 2\sum_{1\le i<j\le m}\left(\mu^{P}_{D_W}\left(q_i q_j\right)\right)^2+\sum_{i=1}^{m} d^{2}_{\mu^{P}_{D_W}\left(q_i q_j\right)}\left(q_i\right)\end{aligned}$$

Therefore, $\sum_{i=1}^{m}\upsilon_i^2 = 2\sum_{1\le i<j\le m}\left(\mu^{P}_{D_W}\left(q_i q_j\right)\right)^2+\sum_{i=1}^{m} d^{2}_{\mu^{P}_{D_W}\left(q_i q_j\right)}\left(q_i\right)$

Similarly, $\sum_{i=1}^{m}\zeta_i^2 = 2\sum_{1\le i<j\le m}\left(\mu^{P}_{D_U}\left(q_i q_j\right)\right)^2+\sum_{i=1}^{m} d^{2}_{\mu^{P}_{D_U}\left(q_i q_j\right)}\left(q_i\right)$,

$$\sum_{i=1}^{m}\tau_i^2 = 2\sum_{1\le i<j\le m}\left(\mu^{N}_{D_W}\left(q_i q_j\right)\right)^2+\sum_{i=1}^{m} d^{2}_{\mu^{N}_{D_W}\left(q_i q_j\right)}\left(q_i\right),$$

$$\sum_{i=1}^{m}\xi_i^2 = 2\sum_{1\le i<j\le m}\left(\mu^{N}_{D_U}\left(q_i q_j\right)\right)^2+\sum_{i=1}^{m} d^{2}_{\mu^{N}_{D_U}\left(q_i q_j\right)}\left(q_i\right)$$

Theorem 3.2. Let G be the FBPG, then the Laplacian matrix of G is denoted by L(BG). If $\upsilon_i \ge \upsilon_i + 1$, $\zeta_i \ge \zeta_{i+1}$, $\tau_i \ge \tau_{i+1}$ and $\xi_i \ge \xi_{i+1}$ where i=1,2,…,m-1 are the eigenvalues of $L\left(\mu^{P}_{D_W}\left(q_i q_j\right)\right), L\left(\mu^{P}_{D_U}\left(q_i q_j\right)\right), L\left(\mu^{N}_{D_W}\left(q_i q_j\right)\right)$ and $L\left(\mu^{N}_{D_U}\left(q_i q_j\right)\right)$ respectively, then

$$\sum_{i=1}^{m}\alpha_i = 0, \sum_{i=1}^{m}\beta_i = 0, \sum_{i=1}^{m}\chi_i = 0 \text{ and } \sum_{i=1}^{m}\rho_i = 0.$$

$$\sum_{i=1}^{m}\alpha_i^2 = 2\left[\sum_{1\le i<j\le m}\left(\mu^{P}_{D_W}\left(q_i q_j\right)\right)^2+\frac{1}{2}\sum_{i=1}^{m}\left(d_{\mu^{P}_{D_W}\left(q_i q_j\right)}\left(q_i\right)-\frac{2\sum_{1\le i<j\le m}\left(\mu^{P}_{D_W}\left(q_i q_j\right)\right)}{m}\right)^2\right],$$

$$\sum_{i=1}^{m}\beta_i^2 = 2\left[\sum_{1\le i<j\le m}\left(\mu^{P}_{D_U}\left(q_i q_j\right)\right)^2+\frac{1}{2}\sum_{i=1}^{m}\left(d_{\mu^{P}_{D_U}\left(q_i q_j\right)}\left(q_i\right)-\frac{2\sum_{1\le i<j\le m}\left(\mu^{P}_{D_U}\left(q_i q_j\right)\right)}{m}\right)^2\right],$$

$$\sum_{i=1}^{m}\chi_i^2 = 2\left[\sum_{1\le i<j\le m}\left(\mu^{N}_{D_W}\left(q_i q_j\right)\right)^2+\frac{1}{2}\sum_{i=1}^{m}\left(d_{\mu^{N}_{D_W}\left(q_i q_j\right)}\left(q_i\right)-\frac{2\sum_{1\le i<j\le m}\left(\mu^{N}_{D_W}\left(q_i q_j\right)\right)}{m}\right)^2\right] \text{ and}$$

$$\sum_{i=1}^{m}\rho_i^2 = 2\left[\sum_{1\le i<j\le m}\left(\mu^{N}_{D_U}\left(q_i q_j\right)\right)^2+\frac{1}{2}\sum_{i=1}^{m}\left(d_{\mu^{N}_{D_U}\left(q_i q_j\right)}\left(q_i\right)-\frac{2\sum_{1\le i<j\le m}\left(\mu^{N}_{D_U}\left(q_i q_j\right)\right)}{m}\right)^2\right]$$

Example 3.1.
From example 2.1

$$D(G)=\begin{bmatrix} (0.5,1.5,-0.6,-1) & (0,0,0,0) & (0,0,0,0) & (0,0,0,0) & (0,0,0,0) \\ (0,0,0,0) & (0.3,1.6,-0.3,-0.9) & (0,0,0,0) & (0,0,0,0) & (0,0,0,0) \\ (0,0,0,0) & (0,0,0,0) & (1,1.8,-0.8,-1.3) & (0,0,0,0) & (0,0,0,0) \\ (0,0,0,0) & (0,0,0,0) & (0,0,0,0) & (1.2,1.8,-1.3,-1.5) & (0,0,0,0) \\ (0,0,0,0) & (0,0,0,0) & (0,0,0,0) & (0,0,0,0) & (1,0.9,-0.6,-0.9) \end{bmatrix}$$

$$L(G)=\begin{bmatrix} (0.5,1.5,-0.6,-1) & (-0.1,-0.8,0.1,0.5) & (0,0,0,0) & (-0.4,-0.7,0.5,0.5) & (0,0,0,0) \\ (-0.1,-0.8,0.1,0.5) & (0.3,1.6,-0.3,-0.9) & (-0.2,-0.8,0.2,0.4) & (0,0,0,0) & (0,0,0,0) \\ (0,0,0,0) & (-0.2,-0.8,0.2,0.4) & (1,1.8,-0.8,-1.3) & (-0.3,-0.6,0.4,0.5) & (-0.5,-0.4,0.2,0.4) \\ (-0.4,-0.7,0.5,0.5) & (0,0,0,0) & (-0.3,-0.6,0.4,0.5) & (1.2,1.8,-1.3,-1.5) & (-0.5,-0.5,0.4,0.5) \\ (0,0,0,0) & (0,0,0,0) & (-0.5,-0.4,0.2,0.4) & (-0.5,-0.5,0.4,0.5) & (1,0.9,-0.6,-0.9) \end{bmatrix}$$

AL(G) = {(0,0,-1.4205,-2.2117),(0.3432,0.8344,-0.9110,-1.597),(0.5, 1.6197, -0.7382,-1.1721),(1.4568, 2.0160,-0.3810,-0.6188),(1.7,3.1299,-0.1494,0)}

EL(G)= (4, 7.6, 3.6046, 5.5996)

CONCLUSION

A PFS method is effective for displaying issues of vulnerability, indeterminacy and conflicting data where humanoid information is fundamental and humanoid assessment required. PF models give more accuracy, adaptability and similarity to the framework when contrasted with the traditional, fuzzy. A PFG can portray the vulnerability of a wide range of organizations. We presented the ideas of E and LE of FBPGs conditions and examined their properties. We calculated upper and lower limits for a FBPG's energy and Laplacian energy.

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