PRIME AND SEMIPRIME IDEALS IN TERNARY Γ-SO-SEMIRINGS-II

Dr.V.Amarendra Babu¹, M.Ankarao², Dr.Bhagyalakshmi Kothuru³,
¹Department of Mathematics, AcharyaNagarjuna University.
²Department of Mathematics(PG), J.K.C College, Guntur-6.
³Department of Mathematics, KKR&KSR Institute of Technology &Sciences Vinjanampadu, Guntur-17

Abstract - “The set of all partial functions over a set under a natural addition, functional composition and functional relation on the, forms a Γ-SO-semiring. The concepts of prime ideal, semi prime ideals in ternary Γ-SO-ring are introduced.”

Index Terms - “prime ideal, semiprime ideal, Ternary Γ-SO semiring.”

1. INTRODUCTION

“In 1981 the concept of Γ - semigroup as generalization of semi group introduced by Sen. H.S Vandiver develops the theory of semi ring in 1934. The notion of Γ- semirings was introduced by M.MuralikrishnaRao in 1995. So some classical notion of ternary Γ-SO semirings are introduced in this paper. In 2019 K.Bhagyalakshmi and Dr.V.AmarendraBabu developed ideal theory in Ternary Γ-SO semirings. In this paper we familiarize the notions of irreducible, strongly irreducible bi-ideals of Ternary Γ-SO semirings and obtain characterizations of prime, semiprime, irreducible and strongly irreducible bi-ideals in regular Ternary Γ-SO semiring. Ternary Γ-SO semiring is denoted by TΓSS, Commutative Ternary Γ-SO semiring is denoted by CTΓSS, prime ideal by PI, semiprime ideal is denoted throughout this paper, minimal prime ideal by MPI.”

2. PREREQUISITES

The following are the prerequisites for this paper.

Definition 2.1: “A be partial Γ-monoid is a triple (R, Γ, Σ) where R, Γ are non-empty sets and Σ is a partial addition defined on some but not necessarily all families (ai : i ∈ I) in R with the following laws:

1) Unary sum axiom: If (ai : i ∈ I) is a one element family in R and I = {j} then \( \sum (a_i : i ∈ I) \) is defined and equal to \( a_j \).
2) Partition Associative axiom: If (ai : i ∈ I) is a family in R and (aj : j ∈ J) is a partition of I, then (ai : i ∈ I) is sum-able if and only if (ai : i ∈ Ij) is sum-able for every j in J, \( \left( \sum (a_i : i ∈ I_j) : j ∈ J \right) \) is sum-able and \( \sum (a_i : i ∈ I_j) = \sum \left( \sum (a_i : i ∈ I_j) : j ∈ J \right) \).

Definition 2.2: “Let M, Γ be partial Γ-monoids then M is said to be partial ternary gamma semiring provided \( \exists \) a mapping \( M × Γ × M × Γ × M → M \) satisfying the following conditions:

1) \( xαyβ(zδpyq) = xα(yδpz)γq = (xαyβz)δpγq \)
2) a family (ai : i ∈ I) is sum-able in M implies that \( (xαyβa_i : \text{for odd } i ∈ I) \) is sum-able in M and \( xαyβ\left[ \sum (a_i : i ∈ I) \right] = \sum (xαyβa_i : \text{for odd } i ∈ I) \)
3) a family (ai : i ∈ I) is sum-able in M implies that \( (xαa_iβy : \text{for odd } i ∈ I) \) is sum-able in M and \( xα\left[ \sum (a_i : i ∈ I) \right]βy = \sum (xαa_iβy : \text{for odd } i ∈ I) \)
4) a family (ai : i ∈ I) is sum-able in M implies that \( (a_iαxβy : \text{for odd } i ∈ I) \) is sum-able in M and \( \left[ \sum (a_i : i ∈ I) \right]αxβy = \sum (a_iαxβy : \text{for odd } i ∈ I) \).
Definition 2.3: "A partial ternary $\Gamma$-semiring said to have a left (lateral, right) unity element provided there exist a family $(e_i : i \in I)$ of $M$ and $(\alpha_i, \beta_i : i \in I)$ of $\Gamma$ such that
\[ \sum_i \alpha_i e_i \beta_i = \alpha \sum_i \alpha_i e_i \beta_i = a \sum_i \alpha e_i \beta_i = a \] for any $a \in M$.

Definition 2.4: "The sum ordering relation $\leq$ in partially ternary $\Gamma$-monoid $M$ is the binary relation such that $a \leq b$ iff there exist an element $c \in M$ such that $b = a + c \forall a, b \in M$.

Definition 2.5: "A sum ordered partially ternary $\Gamma$-monoid (ternary $\Gamma$-so-monoid) in which partial sum ordering is a partial ordering.

Definition 2.6: "A partial ternary $\Gamma$-semiring is said to be sum ordered partially ternary $\Gamma$-semiring (Ternary $\Gamma$-SO-semiring) if the partial $\Gamma$-monoid is SO-$\Gamma$-monoid.

Definition 2.7: "Let $M$ be a partial ternary $\Gamma$-semiring. A non-empty subset of $M$ is said to be left (lateral, right) partial ternary $\Gamma$-ideal of $M$ provided

(i) $(a_i : i \in I)$ is a sum able family of $M$ and $x \in A$ for all $i \in I$ implies $\sum_i x_i \in A$

(ii) for all $x, y \in M$, $z \in A$ implies that $z \alpha x \beta y \in A$, $(x \alpha z \beta) \in A$, $x \alpha y \beta z \in A$.

If $A$ is left, lateral and right partial ternary $\Gamma$-ideal of $M$, then $A$ is called partial ternary $\Gamma$-ideal of $M$.

Definition 2.8: "Let $M$ be a ternary $\Gamma$-SO-semiring. A non-empty subset $A$ of $M$ is said to be a left (lateral, right) ternary $\Gamma$-ideal of $M$, if it satisfies the following:

(i) $A$ is a left (lateral, right) partial ternary $\Gamma$-ideal of $M$.

(ii) $x \in M$ and $y \in A$ such that $x \leq y$ then $x \in A$.

If $A$ is left, lateral as well as right $\Gamma$-ideal of $M$, then $A$ is known as ternary $\Gamma$-ideal of $M$.

Definition 2.9: "Let $M$ be a ternary $\Gamma$-SO-semiring and $A$ be a subset of $M$, then the intersection of all ternary $\Gamma$-ideals containing the set $A$ is called ternary $\Gamma$-ideal generated by $A$ and it is denoted by $(A)$.

Definition 2.10: "A Ternary $\Gamma$-SO-semiring $M$ is said to be complete ternary $\Gamma$-SO-semiring if every family of elements in $M$ is sum able.

Definition 2.11: "A non-empty subset $A$ of a $\Gamma$-SO-ring $R$ is said to be $\Gamma$-sub SO-ring if

(i) $A$ is a sub-SO monoid of $R$

(ii) $A \Gamma A \subseteq A$.

3.0 Prime and semi prime ideals:

Def 3.1: "A proper ideal $P$ of a TTSSM $M$ is called as prime iff for any ideals $R, S, T$ of $M$, $R \Gamma S \Gamma T \subseteq P \Rightarrow R \subseteq P$ or $S \subseteq P$ or $T \subseteq P$.

Example 3.2: "Let $R = [0, 1]$ be the unit interval of real numbers. For any family $(\alpha_i : i \in I)$ in $R$ define $\sum_i \alpha_i = \sup \{ \alpha_i / i \in I \}$ then $R$ is a partial ternary monoid. If we take $\Gamma = \mathbb{W}$ then $R$ is a partial ternary $\Gamma$-monoid. Consider the mapping $(x, \alpha, \beta, y, z) \rightarrow \inf (x, \alpha, \beta, y, z)$ of $R \times \Gamma \times R \times \Gamma \times R \rightarrow R$ then $R$ is a partial ternary $\Gamma$-semiring. Then $R$ is a ternary $\Gamma$-SO-semiring with usual $\leq$ of real numbers. Let $x \in R$. Take $P = [0, x]$.

Let $E, F, G$ are ideals of $R$ such that $E F G \subseteq P$. Then $\exists y, z, w \in R$ such that $E = [0, y], F = [0, z] \& G = [0, w]$. Now $E F G = [0, y][0, z][0, w] = [0, \inf \{ y, a, z, \beta, w \}]$ for every $\alpha, \beta \in \Gamma$ and so, $[0, \inf \{ y, a, z, \beta, w \}] \subseteq P = [0, x]$. Either $y \leq x$ or $z \leq x$ or $w \leq x$. Thus $E = [0, y] \subseteq [0, x] = P$ or $F = [0, z] \subseteq [0, w] = P$ or $G = [0, w] \subseteq [0, x]$. Hence $P = [0, x]$ is a prime ideal of $R$.

Definition 3.3: "A proper ideal $P$ of a TTSSM is called as semiprime if and only if for any ideals $R$ of $M$, $R \Gamma R \Gamma R \subseteq P \Rightarrow R \subseteq P$.

Example 3.4: "Consider the TTSSM, as defined in Example 3.2. Take $P = [0, x]$.

Let $E$ be any ideal of $M$ such that $E F E \subseteq P$. Then $\exists y \in M$ such that $E = [0, y]$. Now $E F E G = [0, y][0, y][0, y] = [0, \inf \{ y, a, z, \beta, w \}]$ for every $\alpha, \beta \in \Gamma$ and so, $[0, \inf \{ y, a, z, \beta, w \}] \subseteq P = [0, x] \Rightarrow y \leq x$. Thus $E = [0, x] \subseteq [0, x] = P$. Hence $P = [0, x]$ is a semiprime ideal of $M$.

Def 3.5: "A subset $A (\neq \emptyset)$ of TTSSM, is entitled an $m$-system if for each $a, b, c \in A$ there exists elements $w_1, w_2, w_3, w_4$ of $M$ such that $a w_1 \beta b w_2 \delta c w_3 \epsilon w_4 d \subseteq A$ or $a a w_1 \beta b w_2 \delta w_3 \epsilon w_4 f \subseteq A$ or $aa w_1 \beta b w_2 \delta w_3 \epsilon w_4 f \subseteq A$. 
Th3.6: “A proper ideal \( L \) of a TTSSM is prime iff its complement \( L' \) is an m-system”.

Pf: Let \( L \) be a PI of a TTSSM \( M \). Suppose \( d,e \notin L \). Then \( d,e \notin L' \).

If possible, let \( L' \) be not an m-system.

Then for all \( \{w_1, w_2, w_3, w_4 \in M \mid daw_1\beta w_1\delta f \notin L \} \) or \( daw_1\beta w_1\delta w_1\eta w_1\phi \notin L' \) or \( daw_1\beta w_2\gamma \delta w_2\eta \phi w_2 \notin L' \) or \( w_1\alpha d \beta w_2\gamma \delta w_2\eta w_2 \phi \notin L' \).

\( \Rightarrow daw_1\beta w_1\delta f \in L \) or \( daw_1\beta w_2\gamma \delta w_2\eta \phi w_2 \in L \) or \( daw_1\beta w_3\gamma \delta w_3\eta \phi w_3 \in L \) or \( w_1\alpha d \beta w_2\gamma \delta w_2\eta w_2 \phi \in L \). 

Thus \( d,e \notin P \) implies that \( d\Gamma d\Gamma \Gamma \delta \Gamma f \subset L' \), \( d\Gamma d\Gamma \Gamma \Gamma \Gamma f \subset L \) and \( M \Gamma d\Gamma d\Gamma \Gamma \Gamma \Gamma f \subset L \).

Hence by known theorem \( L \) is a prime ideal of \( M \).

Def3.7: “Let \( M \) be a TTSS and \( L \) be an ideal of \( M \). Then \( L \) is called a maximal ideal of \( M \) if \( L \neq M \) and there does not exist any other ideal \( I \) of \( M \) such that \( L \subset I \subset M \).”

Th3.8: “Let \( A \) be an m-system and \( N \) an ideal of a TTSSM such that \( N \Gamma \Gamma A = \Phi \). Then there exists a maximal ideal \( L \) of \( M \) containing \( N \) such that \( L \Gamma A = \Phi \). Moreover, \( L \) is also a prime ideal of \( M \).”

Pf: The collection \( I(M) \) of all proper ideals of \( M \) containing \( N \) and each of which has an empty intersection is a partially ordered set \( I(M) \) has an upper bound which is their union. Hence by Zorn’s lemma \( I(M) \) possesses a maximal element \( L \) such that \( L \Gamma I \subset M \).”

If possible, let \( L \) be not a PI of \( M \). Then \( x \notin L, y \notin L \) and \( z \notin L \). Let \( I = L+<x>, J = L+<y> \) and \( K = L+<z> \) be three ideals of \( M \) such that \( I \Gamma J K \subset L \) but \( I \subset L, J \subset L \) and \( K \subset L \). Then we see that all the ideals \( I, J \) and \( K \) properly contain \( L \) and hence by maximality of \( L \), we get \( I \Gamma A \neq \phi \), \( J \Gamma A \neq \phi \) and \( K \Gamma A \neq \phi \). So \( \exists e,f,g \in A \) such that \( e \in I, f \in J \) and \( g \in K \).

Q A is an m-system, \( \exists \) elements \( x_1, x_2, x_3, x_4 \) of \( M \) such that \( eax_1 \beta f x_1 \delta g \in A \) or \( eax_1 \beta x_1 \gamma f \delta x_1 \eta x_1 \phi g \in A \) or \( eax_1 \beta x_1 \gamma f \delta x_1 \eta x_1 \phi g = A \).

If \( eax_1 \beta x_1 \gamma f \delta x_1 \eta x_1 \phi g = eax_1 \beta x_1 \gamma f (\delta x_1 \eta x_1 \phi g) \in I \Gamma J K \subset M \)

Thus any case, we arrive at the contradiction that \( L \Gamma A \neq \Phi \). Hence \( L \) is a PI of \( M \).

Def3.9: A PI \( K \) of a TTSSM \( M \) is called a MPI belonging to an ideal \( I \) of \( M \) if \( I \subset K \) and there exists no other PI \( K' \) of \( M \) such that \( I \subset K' \subset K \).
Th3.10: A PI"P" of a TTSS M is a MPI belonging to an ideal I of M iff its complement $P^c$ is a maximal m-system $\ni P^c I I = \Phi$. 

**Proof:** Let P be a MPI belonging to I. By th.3.6 $P^c$ is an m-system with $P^c \subseteq I'$ so that $P^c I I' I I = \Phi$ i.e. $P^c I I = \Phi$. By applying Zorn’s lemma to the class of m-systems of M, we can find a maximal m-system A such that $A I I' I = \Phi$. Let L be a maximal ideal containing $I \ni L A = \Phi$. Then by Th.3.8, L is also a prime ideal of M and $I \subseteq L \subseteq A^c \subseteq P$. Since P is an MPI containing I, $M = A^c = P$ and hence $A = P^c$.

Conversely, suppose that $P^c$ is a maximal m-system $\ni P^c I M = \Phi$. Then $I \subseteq L \subseteq P$ and hence $P^c \subseteq M^c \subseteq I^c$. Therefore, $L' I I' I = \Phi$ i.e. $L' I I = \Phi$. By virtue of maximality of $P^c$, we have $P^c = L$ & P=L. Thus P is a prime ideal containing I. Now we have to S.T P is a MPI containing I. If possible, let Q be any PI containing I and properly contained in P. Then $P^c I I' I I' I = \Phi$ which is a contradiction.

∴ P is a MPI containing I.

**Corollary3.11:** Every prime ideal containing an ideal I of a TTSS M possesses a MPI belonging to I.

**Proof:** If P is a prime ideal containing I, then there exists a maximal m-system A such that $P^c \subseteq A$. Thus by Th: 3.10 $Q = A^c \subset P$ is a MPI containing I.

Def3.12: A proper ideal P of a TTSS M is called a “completely prime ideal” of M if $d, e, f \in P \Rightarrow d \in P$ or $e \in P$ or $f \in P$ for a few three elements d,e, f of M.

Note3.12: “Every completely prime ideal of M is surely a prime ideal of M but the converse may not be true, in general. However, for a CTΓSS both the concepts coincide.”

Lemma3.13: A proper ideal Q of a TTSS M is “completely prime” iff $Q^c$ is a ternary subsemiring of M.

**Proof:** Suppose $Q^c$ is a ternary subsemiring of M. Let $hai \beta j \in Q$. If possible, let $h, i, j \in Q^c$.

Since $Q^c$ is a ternary subsemiring of M, we have $hai \beta j \in Q^c$, a contradiction. So either $h \in Q$ or $i \in Q$ or $j \in Q$. Consequently, Q is “completely prime”.

By reversing the above argument the converse follows:

Theorem3.14: A proper ideal O of a TTSS M is completely prime iff for each pair of non-negative integers m & n with even sum, $M^{m\Gamma(hai \beta j)\Gamma M^n} \subseteq \Omega$ implies $h \in O$ or $i \in O$ or $j \in O$.

**Proof:** Suppose O is a CPI of M and

$M^{m\Gamma(hai \beta j)\Gamma M^n} \subseteq \Omega$.

Where m and n are non-negative integers with even sum.

Then $(hai \beta j) \Gamma(hai \beta j) \Gamma \Gamma \Gamma \Gamma (m + n + 1)$ times belong to O.

Q O is a CPI of M & by induction, we have $hai \beta j \in O \Rightarrow h \in O$ or $i \in O$ or $i \in O$.

Conversely, suppose that $hai \beta j \in O$. If $m$ & $n$ are non-negative integers with even sum, $M^{m\Gamma(hai \beta j)\Gamma M^n} \subseteq M^{m\Gamma M^n} \subseteq O$. Thus, $h \in O$ or $i \in O$ or $j \in O$.

∴ O is completely prime.

Def3.15: A prime radical of a ideal P of TTSS M is defined as the intersection of all prime ideals of M and is denoted by P(M).

Remark3.16: Since every prime ideal of a TTSS contains 0, $P(S) \neq \phi$.

Theorem3.17: For a TTSS M, $P(M) = \{m \in M: \text{every m-system of M containing m contains zero of M}\}$

**Proof:** Suppose that $P'(S) = \{a \in M: \text{every m-system of M containing 'a' containing zero of M}\}$. Let $b \in P$ for some PI P of M. By theorem3.14, $P^c$ is an m-system $Q 0 \in P, 0 \in P^c$ Thus $P^c$ is an m-system of M containing “b” but not containing zero of M. Consequently, $b \notin P(M)$ and hence we find that $P'(M) \subseteq P(M)$ - - (1)
Again, let \( b \notin P'(M) \).

Then \( \exists \) an m-system \( A \) of \( M \) such that \( b \in A \) but \( 0 \notin A \). Thus by theorem 3.8, \( \exists \) a PI “P” of \( M \), \( \exists P \ A = \phi \).

Consequently, \( b \notin P \) and hence \( b \notin P(M) \).

So we find that \( P(M) \subseteq P'(M) \) ....(2)

From (1) & (2), it follows that \( P'(M) = P(M) \).

**Th3.18:** If \( L \) is an ideal of TTSS \( M \) then \( P(L) = P(M) \), where \( P(L) \) denotes the prime radical of \( L \) considering \( L \) as TTSS.

**Pf:** Let the collection \( \Delta_1 \) be of all prime ideals of \( M \) & \( \Delta_2 \) be the collection of all prime ideals of \( L \). Then by known theorem \( P \in \Delta_1 \) implies that \( P \mid L \in \Delta_2 \).

So \( P(L) = \prod_{Q \in \Delta_1} Q \subseteq \prod_{P \in \Delta_2} (L \mid P) = L \mid (\prod_{P \in \Delta_2} P) = L \mid P(M) \) --- (1)

Now \( P(M) \subseteq L \mid P(L) = P(L) \) ---- (2) from (1) & (2), we get \( P(L) = L \mid P(M) \).

**Def3.19:** The completely prime radical of a TTSS \( M \) is defined as the intersection of all completely prime ideals of \( M \) and is denoted by \( P_c(M) \).

Note: Since every completely prime ideal of a TTSS \( M \) is also a prime ideal of \( M \), \( P(M) \subseteq P_c(M) \).

**Remark3.20:** In a commutative TTSS \( M \), prime ideals of \( M \) coincide with its completely prime ideals. Hence for a commutative TTSS \( M \), we find that \( P(M) = P_c(M) \).

**Def3.21:** \( AT \) TTSS is called a prime TTSS if the zero ideal is a prime ideal of \( M \).

“Remark: It is to be noted here that each ternary ring ideal of a ternary ring \( T \) is also a ternary semiring ideal: however, the converse need not be true, in general. Thus if a ternary ring \( M \) is a prime ternary ring then the ternary semiring \( M \) is a prime ternary semiring.”

**Def3.21:** A proper ideal \( Q \) of a TTSS \( M \) is called a semiprime ideal of \( M \) if \( J \mid J \subseteq Q \) implies \( J \subseteq Q \) for any ideal \( J \) of \( M \).

**Note3.22:** Every PI of a TTSS \( M \) is also a SPI of \( M \).

**Th3.23:** \( \text{“A necessary and sufficient condition for an element ‘s’ of a TTSS N to belong to a semi prime ideal R of N is that } N \Gamma s \Gamma N \subseteq R \text{”} \).

**Pf:** Assume \( R \) is a SPI of \( N \).

If \( s \in R \), then clearly \( N \Gamma s \Gamma N \subseteq N \Gamma R \Gamma N \subseteq R \).

Conversely, let \( N \Gamma s \Gamma N \subseteq R \).

Then \( N \Gamma T \Gamma s \Gamma T \Gamma N \subseteq N \Gamma R \Gamma N \subseteq R \).

Now \( \langle s \rangle \Gamma \langle s \rangle \Gamma \subseteq (N \Gamma N \Gamma s + s \Gamma N \Gamma N + N \Gamma s \Gamma N + n \Gamma s) \Gamma \)

\( (N \Gamma T \Gamma s + s \Gamma T \Gamma N + N \Gamma T \Gamma s \Gamma T \Gamma N + n \Gamma s) \Gamma \)

\( (N \Gamma T \Gamma s + s \Gamma T \Gamma N + N \Gamma s \Gamma N + n \Gamma s) \subseteq (N \Gamma s \Gamma N + N \Gamma T \Gamma s \Gamma T \Gamma N) \subseteq R \)

is semiprime, we have \( \langle s \rangle \subseteq R \) \& \( s \in R \).

The resulting Th. gives a description of a semiprime ideal in a TTSS \( N \).

**Th 3.24:** Let \( M \) be a TTSS then a proper ideal \( Q \) of \( M \) then the conditions are equivalent.

(i) \( Q \) is semiprime

(ii) \( s \Gamma M \Gamma s \Gamma M_1 \Gamma s \subseteq Q, s \Gamma M \Gamma M_1 \Gamma M_1 \Gamma M_1 \Gamma s \subseteq Q, s \Gamma M_1 \Gamma M_1 \Gamma M_1 \Gamma M_1 \Gamma M_1 \subseteq Q \& \\
M_1 \Gamma s \Gamma M \Gamma s \Gamma M_1 \Gamma s \subseteq Q \Rightarrow s \in Q \).

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\[(\text{iii}) \quad \langle s, g \rangle \langle s, g \rangle \langle s, g \rangle \subseteq Q \Rightarrow s \in Q. \]

**Proof:**
(i) Assume \( s \) is a Pi of \( M \)
\& \( s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \subseteq Q, \ s' \Gamma s \Gamma s' \Gamma s' \Gamma s' \subseteq Q, \ s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \subseteq Q, \ s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \subseteq Q.

Then \( (M \Gamma s' \Gamma M) \Gamma (M \Gamma s' \Gamma M) \Gamma (M \Gamma s' \Gamma M) = M \Gamma (s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma M \subseteq M \Gamma Q \Gamma M \subseteq Q.

\[(M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') = (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (s' \Gamma s' \Gamma s' \Gamma s' \Gamma s').

\[(M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \subseteq M \Gamma (s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma M \subseteq M \Gamma Q \Gamma M \subseteq Q.

\[(M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') = (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s').

\[(M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') = (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \subseteq M \Gamma Q \Gamma M \subseteq Q.

\[(M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \subseteq M \Gamma (s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma M \subseteq M \Gamma Q \Gamma M \subseteq Q.

\[(M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \subseteq M \Gamma (s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma M \subseteq M \Gamma Q \Gamma M \subseteq Q.

\[(M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \subseteq M \Gamma (s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma M \subseteq M \Gamma Q \Gamma M \subseteq Q.

\[(M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \subseteq M \Gamma (s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma M \subseteq M \Gamma Q \Gamma M \subseteq Q.

\[(M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \subseteq M \Gamma (s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma M \subseteq M \Gamma Q \Gamma M \subseteq Q.

\[(M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \subseteq M \Gamma (s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma M \subseteq M \Gamma Q \Gamma M \subseteq Q.

\[(M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \subseteq M \Gamma (s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma M \subseteq M \Gamma Q \Gamma M \subseteq Q.

\[(M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \subseteq M \Gamma (s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma M \subseteq M \Gamma Q \Gamma M \subseteq Q.

\[(M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \subseteq M \Gamma (s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma M \subseteq M \Gamma Q \Gamma M \subseteq Q.

\[(M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \subseteq M \Gamma (s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma M \subseteq M \Gamma Q \Gamma M \subseteq Q.

\[(M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \subseteq M \Gamma (s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma M \subseteq M \Gamma Q \Gamma M \subseteq Q.

\[(M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \subseteq M \Gamma (s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma M \subseteq M \Gamma Q \Gamma M \subseteq Q.

\[(M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \subseteq M \Gamma (s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma M \subseteq M \Gamma Q \Gamma M \subseteq Q.

\[(M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \subseteq M \Gamma (s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma M \subseteq M \Gamma Q \Gamma M \subseteq Q.

The above each component are ideals of \( M \), by primeness of \( Q \), it shows that
\[(M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma (M \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \subseteq M \Gamma (s' \Gamma s' \Gamma s' \Gamma s' \Gamma s' \Gamma s') \Gamma M \subseteq M \Gamma Q \Gamma M \subseteq Q.

\[Q \quad \text{is a semiprime ideals of} \quad M, \quad \langle s, g \rangle \langle s, g \rangle \langle s, g \rangle \subseteq Q \& \text{thus} \quad s \in Q.

In the same manner if \( (M \Gamma s' \Gamma M) \subseteq Q \) then \( s \in Q \).

\[Q \quad \text{is a semiprime ideals of} \quad M, \quad \langle s, g \rangle \langle s, g \rangle \subseteq Q \& \text{thus} \quad s \in Q.

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Assume (ii) \( s \in M \) for some \( s \in M \).

\[ \sum_{\substack{\text{all elements } m_i, n_i \geq 0 \text{ with } m_i + n_i = 0 \in \mathbb{Z} \\text{ for } i = 1, \ldots, k \\text{ and } \sum m_i \leq \infty \text{ or } \sum n_i \leq \infty \}} \]

\[ s \Gamma M \Gamma s \Gamma M = s \Gamma (M \Gamma s \Gamma M) \Gamma s \subseteq \langle s \rangle \Gamma \subseteq Q \]

\[ s \Gamma M \Gamma s \Gamma M \Gamma s = s \Gamma (M \Gamma s \Gamma M) \Gamma s \subseteq \langle s \rangle \Gamma \subseteq Q \]

By (ii) \( s \in Q \).

(iii) \( \Rightarrow \) (i)

Suppose the condition (iii) holds and \( J \Gamma J \Gamma J \subseteq Q \) for an ideal \( J \) of \( M \).

If possible \( J \not\subseteq Q \). Then \( \exists \) an element \( s \in J \) \( \forall s \not\in J \).

Now \( s \Gamma s \Gamma s \Gamma s \Gamma s \subseteq J \). This implies that \( s \in Q \), a contradiction.

Thus \( J \subseteq Q \) & hence \( Q \) is a SPI of \( M \).

**Corollary 3.25:** A proper ideal \( Q \) of a CT**\( T \) SS \( M \) is semiprimeiff \( s \alpha s \beta s \in Q \) implies that \( s \in Q \) for any element \( s \) of \( S \).

**Def 3.26:** A subset \( D (\neq \emptyset) \) of a TT**\( S \) SS \( M \) is so-called a p-system if for each \( d \in D \) \( \exists \) elements \( s_1, s_2, s_3, s_4 \) of \( M \) & \( \alpha, \beta, \gamma, \delta, \epsilon \in \Gamma \) such that \( d \alpha s_1 \beta s_2 \gamma \delta \epsilon d \in D \) or \( d \alpha s_1 \beta s_2 \gamma \delta \delta s_3 \eta \delta \epsilon s_4 \in D \) or \( s_1 \alpha d \beta s_2 \gamma \delta \delta s_3 \eta \delta \epsilon s_4 \in D \).

**Def 3.27:** A proper ideal \( Q \) of a TT**\( S \) SS \( M \) is called a completely semiprime ideal of \( M \) if \( s \alpha s s \alpha s \in Q \) implies that \( s \in Q \).

**Th 3.28:** A proper ideal \( Q \) of a TT**\( S \) SS \( M \) is semiprimeiff \( Q^c \) (complement of \( Q \)) is a p-system.

**Th 3.29:** Let \( D \) be a p-system and \( J \) be an ideal of a TT**\( S \) SS \( M \) \( \exists D \Gamma J = \emptyset \). Then \( \exists \) a maximal ideal \( N \) of \( M \) containing \( J \) \( \ni D \Gamma J = \emptyset \). Moreover, \( N \) is also a semiprime ideal of \( M \).

**Th 3.30** A proper ideal \( Q \) of a TT**\( S \) SS \( M \) is a minimal semiprime ideal belonging to the ideal \( J \) iff its complement \( Q^c \) is a maximal p-system such that \( Q^c \Gamma J = \emptyset \).

**Corollary 3.31:** There exist a unique minimal semiprime ideal belonging to an ideal \( J \) of \( M \), namely, the intersection of all semiprime ideals containing \( J \).

**Corollary 3.32:** There exists a unique maximal p-system which does not intersect any ideal \( J \) of a TT**\( S \) SS \( M \).

**Theorem 3.33:** A proper ideal \( Q \) of a TT**\( S \) SS \( M \) is completely semiprime iff for each pair with even sum of non-negative integers \( m \) & \( n \), \( S^n \Gamma (s \alpha s s \alpha s) \Gamma S^n \subseteq Q \) implies \( s \in Q \).

**Remark:** Every m-system is a p-system and it is clear that the union of p-systems is again a p-system but not conversely.

**Th 3.34:** A subset \( D (\neq \emptyset) \) of a TT**\( S \) SS \( M \) is p-system iff it is the union of m-systems.

**Pf:** Suppose \( D \) is a union of m-systems. Then \( D \) is a p-system, since any m-system is also a p-system & the union of p-systems is a p-system, by above Remark.

Conversely, Let \( D \) be a p-system and \( s_0 \in D \). Now we set \( C_0 = \{ s_0 \} \) \( C_{k+1} = \{ s : s = a a t_1 t_2 c \in C_{k+1} \} \) for some \( a, b, c \in \bigcup_{j=0}^{k} C_j \), \( t_1, t_2, t_3, t_4 \in M \) & \( \alpha, \beta, \gamma, \delta, \epsilon \in \Gamma \).
Next we show that \( C = \bigcup_{k=0}^{\infty} C_k \) is a subset of \( D \).

To show this we shall have to show that if \( s \in C \), then \( s = s_0 \) or \( s = s_1 \alpha r_1 \beta s_0 \gamma r_1 \epsilon s_0 \) or \( s = s_2 \alpha r_2 \beta r_2 y s_0 \delta r_2 \epsilon s_0 \eta r_2 \) or \( s = s_3 \alpha r_3 \beta r_3 y s_0 \delta r_3 \epsilon s_0 \eta r_3 \) or \( s = r_4 s_0 \beta r_4 y s_0 \delta r_4 \epsilon s_0 \eta r_4 \) or \( s = s_0 \alpha r_0 \beta s_0 \gamma r_0 \epsilon s_0 \) or \( s = s_1 \alpha r_1 \beta s_1 \gamma r_1 \epsilon s_1 \) or \( s = s_2 \alpha r_2 \beta r_2 y s_2 \delta r_2 \epsilon s_2 \eta r_2 \) or \( s = s_3 \alpha r_3 \beta r_3 y s_3 \delta r_3 \epsilon s_3 \eta r_3 \) or \( s = r_4 s_4 \beta r_4 y s_4 \delta r_4 \epsilon s_4 \eta r_4 \) for some \( r_0, r_1, r_2, r_3, r_4 \in M, \alpha, \beta, \gamma, \delta, \epsilon, \eta \in \Gamma \).

We shall show this by using induction on \( k \). If \( s \in C_0 \), then clearly; \( s = s_0 \in D \).

Suppose the result is true for all \( j \leq k \). Now if \( s \in C_{k+1} \), then \( \exists a, b, c \in \bigcup_{j=0}^{k} C_j \exists s = a a t_1 b \beta \gamma t_2 \delta c \) or \( s = a a t_1 b \beta t_2 \gamma b \delta t_3 \epsilon \eta c \eta \) or \( s = a a t_1 b \beta t_2 \gamma b \delta t_3 \epsilon \eta c \eta \) for some \( t_1, t_2, t_3, t_4 \in M, \alpha, \beta, \gamma, \delta, \epsilon, \eta \in \Gamma \).

Now by hypothesis of induction, we have \( a = s_0 \alpha u_0 \beta s_0 \gamma u_0 \delta s_0 \) or \( a = s_0 \alpha u_1 \beta u_1 \gamma s_0 \delta u_1 \epsilon s_0 \eta u_0 \eta u_1 \) or \( a = s_0 \alpha u_2 \beta u_2 \gamma s_0 \delta u_2 \epsilon s_0 \eta u_0 \eta u_2 \) or \( a = s_0 \alpha u_3 \beta u_3 \gamma s_0 \delta u_3 \epsilon s_0 \eta u_0 \eta u_3 \) or \( a = s_0 \alpha u_4 \beta u_4 \gamma s_0 \delta u_4 \epsilon s_0 \eta u_0 \eta u_4 \) for some \( u_1, u_2, u_3, u_4 \in M, \alpha, \beta, \gamma, \delta, \epsilon, \eta \in \Gamma \).

Thus by considering each of the 4 cases the result follows because if we consider \( s = a a t_1 b \beta \delta t_2 \epsilon c, a = s_0 \alpha u_0 \beta u_0 \gamma s_0 \delta u_0 \epsilon s_0 \eta u_0 \eta u_0 \)

\[
b = s_0 \alpha v_0 \beta v_0 \gamma s_0 \delta v_0 \epsilon s_0 \eta v_0 \eta v_0 \quad \text{or} \quad b = s_0 \alpha v_1 \beta v_1 \gamma s_0 \delta v_1 \epsilon s_0 \eta v_0 \eta v_1 \quad \text{or} \quad b = s_0 \alpha v_2 \beta v_2 \gamma s_0 \delta v_2 \epsilon s_0 \eta v_0 \eta v_2 \quad \text{or} \quad b = s_0 \alpha v_3 \beta v_3 \gamma s_0 \delta v_3 \epsilon s_0 \eta v_0 \eta v_3 \]

\[
c = s_0 \alpha w_0 \beta w_0 \gamma s_0 \delta w_0 \epsilon s_0 \eta w_0 \quad \text{or} \quad c = s_0 \alpha w_1 \beta w_1 \gamma s_0 \delta w_1 \epsilon s_0 \eta w_1 \quad \text{or} \quad c = s_0 \alpha w_2 \beta w_2 \gamma s_0 \delta w_2 \epsilon s_0 \eta w_2 \quad \text{or} \quad c = s_0 \alpha w_3 \beta w_3 \gamma s_0 \delta w_3 \epsilon s_0 \eta w_3 \]

\[
da = s_0 \alpha r_0 \beta r_0 \gamma s_0 \delta r_0 \epsilon s_0 \eta r_0 \quad \text{or} \quad d = s_0 \alpha r_1 \beta r_1 \gamma s_0 \delta r_1 \epsilon s_0 \eta r_1 \quad \text{or} \quad d = s_0 \alpha r_2 \beta r_2 \gamma s_0 \delta r_2 \epsilon s_0 \eta r_2 \quad \text{or} \quad d = s_0 \alpha r_3 \beta r_3 \gamma s_0 \delta r_3 \epsilon s_0 \eta r_3 \]

Hence \( C = \bigcup_{k=0}^{\infty} C_k \) is a subset of \( D \).

Now it remains show that \( C = \bigcup_{k=0}^{\infty} C_k \) is an m-system.

To show this let \( a, b, c \) are belongs to \( C \).

Then \( a, b, c \in C_k \) for some natural number \( k \).

Consequently, \( s = a a t_1 b \beta \gamma t_2 \delta c \in C_{k+1} \subseteq C \) or \( s = a a t_1 b \beta t_2 \gamma b \delta t_3 \epsilon \eta c \eta \in C_{k+1} \subseteq C \) or \( s = t_1 a a a t_2 \gamma b \delta t_3 \epsilon \eta c \eta \) for some \( t_1, t_2, t_3, t_4 \in M, \alpha, \beta, \gamma, \delta, \epsilon, \eta \in \Gamma \).

Thus C is an “m-system”.

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Theorem 3.35: A proper ideal $Q$ of a TΓSS M is semiprime if and only if the complement $Q^c$ is the union of m-system of M.

Def 3.36: The prime radical $P(J)$ of the ideal $J$ in a TΓSS M is the intersection of all prime ideals in M which contain J.

We have the following characterization of prime radical of an ideal in a TΓSS M.

Th 3.37: For a proper ideal $J$ of TΓSS M,

$$P(J)=\{r \in M: \text{every m-system in } M \text{ which contains } r \text{ has a non-empty intersection with } J\}.$$ 

Pf: Suppose $P(J)=\{r \in M: \text{every m-system in } M \text{ which contains } r \text{ has a non-empty intersection with } J\}$.

Let $a \notin P^c$. 

Then by Th.3.8, $\exists$ a prime ideal $P$ of M containing $J$ such that $P^c = J$. This implies that $a \notin P$ and hence $a \notin P(J)$ consequently, $P(J) \subset P^c(J)$.

Again, let $b \notin P(J)$.

Then $\exists$ a prime ideal $P$ of M containing $b$ but having an empty intersection with $J$.

Thus $b \notin P^c(J)$ and hence $P^c(J) \subset P(J)$.

Thus it follows that $P^c(J) = P(J)$ and this completes the proof of the theorem.

Th 3.38: An ideal $J$ of a TΓSS M is semiprime if and only if $P(J)=J$.

Pf: Let $J$ be a semiprime ideal of TΓSS M. Then by Th.3.8 the complement $J^c$ is a p-system.

Hence by Th.3.34 $I^c = \bigcup_{i \in \Delta} B_i$, where each $B_i$ is an m-system contained in $I^c$.

$Q I B_i = \phi$ for each $i \in \Delta$, it follows from Th.3.8 that $\exists$ a maximal ideal $P_i$ containing $J$ such that $B_i \cup P_i = \phi$ for each $i \in \Delta$, and each $P_i$ is a prime ideal of M.

Thus $J \subset \bigcap_{i \in \Delta} P_i \subset \bigcap_{i \in \Delta} B_i = (\bigcup_{i \in \Delta} B_i)^c = (J^c)^c = J$.

Consequently, $J = \bigcap_{i \in \Delta} P_i = P(J)$, where $\Delta'$ is the set of all prime ideals of M containing J.

Conversely, suppose that $J=P(J)$. Then $J = \bigcap_{i \in \Delta} \{P: \text{P is a prime ideal of M containing J}\}$.

$J^c = \bigcup_{i \in \Delta} \{P^c: \text{P is a prime ideal of M containing J}\}$.

Since P is a prime ideal of M, by Th. 3.8, it follows that $P^c$ is an m-system and hence by using Th. 3.6 we get $J^c$ is a p-system.

Consequently, $J$ is a SPI of M, by using Th. 3.8

Corollary: If $J$ is an ideal of a TΓSS M, then $P(J)$ is the smallest semiprime ideal of M contains J.

Def 3.39: A proper ideal $J$ of TΓSS M is known to be weakly irreducible if for ideals $K, L, N$ of M, $K I L I N = J \Rightarrow J=K$ or $J=L$ or $J=N$.

The term simply use irreducible to mean weakly irreducible.

Def 3.40: A proper ideal $J$ of TΓSS M is said to be weakly reducible if for ideals $K, L, N$ of M, $K I L I N \subset J \Rightarrow K \subset J$ or $L \subset J$ or $N \subset J$.

Note 3.41: “It is be noted here that a strongly irreducible ideal of a TΓSS M, is an irreducible ideal of M”.

Def 3.42: A non-empty subset $A$ of a TΓSS M, is named an “i-system” if $l, m, n \in A$ implies $<l > I <m > I < n > I A \neq \phi$.
Th3.43: The conditions followed in a TΓSS M, are equivalent:

(i) \( J \) is a strongly irreducible ideal of \( M \)
(ii) If for \( l,m,n \in M : < l > I < m > I < n > \subseteq J \) then \( l \in J \) or \( m \in J \) or \( n \in J \)
(iii) The complement of \( J \), i.e., \( J^c \) is an i-system.

**Pf:** (i) \( \Rightarrow \) (ii) This is a consequence of the Def. 2.2.24

(ii) \( \Rightarrow \) (iii) If possible let \( l,m,n \in J^c \) & \( < l > I < m > I < n > I J^c = \emptyset \).

Then \( < l > I < m > I < n > I J^c = \emptyset \) implies that \( < l > I < m > I < n > \subseteq J \) and hence by using (ii), we have \( l \in J \) or \( m \in J \) or \( n \in J \), which is a contradiction. Consequently \( < l > I < m > I < n > I J^c \neq \emptyset \) & hence \( J^c \) is an i-system.

(iii) \( \Rightarrow \) (i) Let \( E, F, G \) be three ideals of \( M \ni E \subset J, F \subset J & G \subset J \).

Then \( \exists l \in E - J, m \in F - J & n \in G - J \).

Now from (iii), it follows that \( < l > I < m > I < n > I J^c \neq \emptyset \)

ie., there exists an element \( o \in (< l > I < m > I < n >) - J \).

\[ o \in E \cup F \cup G \& o \notin J. \]

\[ \vdash E \cup F \cup G \subset J. \quad \Rightarrow J \text{ is strongly irreducible.} \]

**Lemma3.45:** Let “l” be a non-zero element of a TΓSS M and let \( J \) be a proper ideal of \( M \) not containing ‘l’. Then there exists an irreducible \( H \) of \( M \) containing \( J \) and not containing “l”.

**Th3.46:** Any proper ideal \( J \) of a TΓSS M is the intersection of all irreducible ideals containing it.

**REFERENCES**


**AUTHOR INFORMATION**

**Dr.V.Amarendra Babu** completed his Ph.D. from Acharya Nagarjuna University, Andhra Pradesh, India. He is working as an Assistant Professor in the Department of Mathematics, Acharya Nagarjuna University. In his guidance 7 Ph.D.’s and 3 M.Phil’s, were awarded. He published more than 65 research papers in popular international journals to his credit. His areas of interests are algebra and topology.

**M.Anka Rao** working as a Lecturer in the Department of Mathematics, J.K.C College. He is pursuing his Ph.D. under guidance of Dr.V.Amarendra Babu from Acharya Nagarjuna University, Andhra Pradesh, India. He published more than 3 research papers in popular international journals to his credit. His areas of interests are algebra and topology.

**Dr. Bhagyalakshmi Kothuru** completed her Ph.D. from Acharya Nagarjuna University, Andhra Pradesh, India. She is working as an Associate Professor in the Department of Mathematics, KKR & KSR Institute of Technology & Sciences. She published more than 20 research papers in popular international journals to her credit. Her areas of interests are algebra and topology.