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Piecewise Convex $L_p, 0 < p < 1$, Approximation

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Abstract

Many papers introduced to constrained approximation of convex continuous function by algebraic polynomials, here we approximate the convex function in L_p, $0 < p < 1$, quasi normed spaces using piecewise algebraic polynomials. Also we introduce some properties of these polynomials.

Keywords : $L_p$space , piecewise , convex approximation ,derivative

1 Introduction

The accompanying old style Timan-Freud-Brudnyi Jackson type inequality for the approximation by algebraic polynomials [1, theorem 8-5-3]). Explains the request for the approximation turns out to be fundamentally better closed to the end point of [-1, 1]:

If $\kappa \in \mathrm{N}$, $\mathfrak{r} \in \mathrm{N}_0$ and $\mathcal{F} \in \mathrm{C}^{\mathfrak{r}}$,then for all $\mathfrak{n} \geq \kappa + \mathfrak{r} - 1$, There is polynomial $\mathcal{P}_{\mathfrak{n}} \in \prod_{\mathfrak{n}}$ Satisfy

$$|\mathcal{F}(x) - \mathcal{P}_{\mathfrak{n}}(x)| \leq c(\kappa, \mathfrak{r})\Omega_{\mathfrak{n}}^{\mathfrak{r}}(x)w_k(\mathcal{F}^{\mathfrak{r}}, \Omega_{\mathfrak{n}}(x)), \quad x\epsilon[-1,1] \qquad (1.2)$$

obviously, if we want to have interpolating approximation at the endpoints for f and it's derivatives. We get better approximation degree.

As a direct consequence of corollary (2.3.4) in [2], we obtain the following Telyakovskii-gopengauze version theorem, for the literature review of the subject.

<u>Theorem</u>1.3

In [2-Corollary 2-3-4]) $\mathcal{L}$et $\mathfrak{r}\epsilon\mathrm{N}_0, \kappa \epsilon \mathrm{N}$ and $\mathcal{F} \in \mathrm{C}^{\mathfrak{r}}$,then for any $\mathfrak{n} \geq max\{\kappa + \mathfrak{r} - 1, 2\mathfrak{r} + 1\}$,there is a polynomial $\mathcal{P}_{\mathfrak{n}} \in \Pi_{\mathfrak{n}}$such that (1.2) Is valid and , more over

$$|\mathcal{F}(x) - \mathcal{P}_{\mathfrak{n}}(x)| \leq \mathrm{c}(\mathfrak{r}, \kappa)\vartheta^{2\mathfrak{r}}(x)w_{\mathrm{k}}\left(\mathcal{F}, \vartheta^{\frac{2}{\kappa}}(x)\mathfrak{n}^{-\frac{2(\kappa-1)}{\kappa}}\right), \text{if } 1 - \mathfrak{n}^{-2} \leq |x| \leq 1 \qquad (1.4)$$

We get from theorem 3 in [2], that

If $\gamma \in \mathrm{R}$, $\vartheta^{\frac{2}{\kappa}}(x)\mathfrak{n}^{-\frac{2(\kappa-1)}{\kappa}}$ in (1.4) Cannot be replaced by $\vartheta^{2\mathcal{G}}(x)\mathfrak{n}^{\gamma}$with $\mathcal{G} > 1/\kappa$. therefore, the estimate (1.4) offers the best approximation rate near the end point of closed interval [-1,1].

Now we have a question : Is the above inequalities true in the shape preserving approximation of q-monotone functions ?

The answer is that it is not true for an $\mathfrak{r}$ and κ ,even if we choose n depends on the target function f.This case if we have$1 \leq q \leq 3$, $0 \leq \mathfrak{r} \leq q - 1$ and $\mathfrak{r} + \mathrm{k} \geq q + 2$ in [3] if q=1 in [4] if q=2or q=3) and $q > 4$ and $\mathfrak{r} + \kappa > 3$ in [5].

More finished, for any q, $\mathfrak{r}$, κ, $\mathfrak{n} \in \mathrm{N}$, There exists a function $\mathcal{F}_{\mathfrak{n}} \in \mathrm{C}^{\mathfrak{r}} \cap \Delta^{(q)}$such that (1.4) is not valid for any polynomial $\mathcal{P}_{\mathfrak{n}} \in \prod_{\mathfrak{n}} \cap \Delta^{(q)}$

the development of such an $\mathcal{F}_{\mathfrak{n}}$ is the same as in ([6] ,view also [7,8,9]) .

This imply that, in the case $\mathfrak{r} \geq 1$,(1.4) cannot be true for each function$\mathcal{F} \in \mathrm{C}^{\mathfrak{r}} \cap \Delta^{(q)}$and all $\mathfrak{n} \geq \mathrm{N}(\kappa, \mathfrak{r}, q)$.

We accentuate that this not imply that for each fixed $\mathcal{F} \in \mathrm{C}^{\mathfrak{r}} \cap \Delta^{(q)}$(1.4) is in true for adequately $q = \kappa = 2$ large $\mathfrak{n}$. i.e.(1.4) may still be true if $\mathfrak{n} \geq \mathrm{N}(\mathcal{F})$.(is the principle result of this subject).

The case is different if we have $\kappa$is small and $\mathfrak{r}$ equal to zero, and$q, \mathfrak{n}$ are naturals.(1.2) and (1.4) are satisfied for $\kappa = 1$, which is follows from the case $\kappa = 2$, for q-monotone approximation.

Surely, from the interpolator estimate follow, [11,12](q=2), [10]($q = 1$) and [13]($q \geq 3$):for any $q, \mathfrak{n} \in \mathrm{N}$ and $\mathcal{F} \in \mathrm{C} \cap \Delta^{(q)}$, there exist a polynomial $\mathcal{P}_{\mathfrak{n}} \in \Pi_{\mathfrak{n}} \cap \Delta^{(q)}$Such that

$$|\mathcal{F}(x) - \mathcal{P}_{\mathfrak{n}}(x)| \leq c(q)w_2\left(\mathcal{F}, \frac{\vartheta(x)}{\mathfrak{n}}\right), x\epsilon[-1,1] \qquad (1.5)$$

Where $c > 0$. If we have $\mathfrak{n} \geq 2$ the estimates (1.2) and (1.4) are satisfied for$q = 2$ when $\mathfrak{r}$ equal to zero and κ equal to 3, we find the result in [2], and case $q = 3$, $\mathfrak{r} = 0$ and $\kappa = 3$ or $\kappa = 4$ is no solution has yet been found(Actually unknown if(1.2)holds if $(q, \mathfrak{r}, \kappa) = (3,0,4)$.

Finally, I was able to show in [8] that (2.1) and (2.3) hold if $\mathfrak{r} \in N$, κ=2 and $\mathfrak{n} \geq N(\mathcal{F},\mathfrak{r})$for monotone approximation ($q = 1$) .

define $L_p(I) = \{\mathcal{F}: I \to R : f \in L_p\}$,where I is closed interval between -1,1. and $L^r{}_p(I) = \{\mathcal{F}: I \to R : \mathcal{F}^{\,r} \in L_p\}$

$\|\mathcal{F}\|_{Lp} = (\int_{-1}^{1}|\mathcal{F}(x)|^p)^{\frac{1}{p}}$. For κϵN and interval I ,

$\Delta_{\mathfrak{u}}^{\kappa}(\mathcal{F},x,I) := \sum_{i=0}^{\kappa}(-1)^i\binom{\kappa}{i}f\left(x + \left(\frac{\kappa}{2} - i\right)\mathfrak{u}\right)$, if $x \mp \frac{\kappa\mathfrak{u}}{2} \in I$ and $:= 0$ otherwise.

$w_\kappa(\mathcal{F},t,I) := \sup_{0<\mathfrak{u}<t}\|\Delta_{\mathfrak{u}}^{\kappa}(\mathcal{F},.;I)\|_p$is a measure of the soomthness modulus of f on I. $w_\kappa(\mathcal{F},t) := w_\kappa(\mathcal{F},t,I), L_p^r = L_p^r(I)$, for any interval I we write $w_\kappa(\mathcal{F},\delta,I)$..

We use $\vartheta(x) = \sqrt{1+x^2}$ and $\Omega_{\mathfrak{n}}(x) = \vartheta(x)\mathfrak{n}^{-1} + \mathfrak{n}^{-2}, \mathfrak{n}\epsilon N, \Omega_0 \equiv 1$

$\Pi_{\mathfrak{n}}$ symbolizes the space of algebraic polynomial of degree$\leq \mathfrak{n}$.

A function $\mathcal{F}: [a,b] \to R$ is said to be κ-monotone, $\kappa \geq 1$ on $[a,b]$if and only if for all choices of $\kappa + 1$ distinct points $x_0, x_1, \ldots, x_k$in $[a,b]$the inequality $\mathcal{F}[x_0, x_1, \ldots, x_k] > 0$ holds, where $\mathcal{F}[x_0, x_1, \ldots, x_k] = \sum_{j=0}^{\kappa}\frac{\mathcal{F}(x_j)}{w'(x_j)}$

$I_\iota := I_{\iota,n} := [x_\iota, x_{\iota-1}]$, $h_\iota := h_{\iota,n} := |I_{\iota,n}| = x_{\iota-1} - x_\iota$

$I_{i,\iota} := \bigcup_{\kappa=min\{i,\iota\}}^{max\{i,\iota\}} I_\kappa = [x_{max\{i,\iota\}}, x_{min\{i,\iota\}-1}\}, 1 \leq i,\iota \leq n$

(the shortest interval containing both $I_i$and I_ι)

$x_\iota := x_{\iota,i} := \cos\left(\frac{\iota\pi}{n}\right), 0 \leq \iota \leq n$, 1for $\iota < 0$and $-1$for $\iota > n$ (chebyshev knots)

$h_{i,\iota} := |I_{i,\iota}| = \sum_{\kappa=min\{i,\iota\}}^{max\{i,\iota\}} h_\kappa = x_{min\{i,\iota\}-1} - x_{max\{i,\iota\}}$

$\mathcal{T}_j := \mathcal{T}(x) := \frac{|I_j|}{(|x - x_j| + |I_j|)}, \delta_{\mathfrak{n}}(x) := min\{1, \mathfrak{n}\vartheta(x)\}$

$\Phi^\kappa := \{\mathcal{T} \in c[0,\infty] | \mathcal{T}\uparrow, \mathcal{T}(0) = 0$ and $t_2{}^{-\kappa}\mathcal{T}(t_2) \leq t_1{}^{-\kappa}\mathcal{T}(t_1)$ for $o \leq t_1 \leq t_2\}$. Note: if $\mathcal{F} \in L_p^r$,then

$\Gamma(t) := t^r w_\kappa(\mathcal{F}^{(r)}, t)_p$ is equivalent to afunction from $\Phi^{\kappa+r}$

$\sum_\kappa := \sum_{\kappa,n}$ denoted the $x_j, 1 \leq j \leq n-1$ piecewise polynomials of degree not exceeding $\kappa - 1$, that are continuous.

$\sum_\kappa^{(1)} = \sum_{\kappa,n}^{(1)}$ denoted the set of all $x_j, 1 \leq j \leq n-1$ piecewise polynomials that have continuous derivatives

$\mathcal{P}_j := \mathcal{P}_j(\mathbb{s}) := \mathbb{S}|I_j, 1 \leq j \leq n$ ($\mathbb{S}$ is piecewise polynomials, of pieces $\mathcal{P}_j(x), x \in I_j, 1 \leq j \leq n-1$, and write $\mathbb{S}|I_j$).

$\mathfrak{b}_{i,j}(\mathbb{s},\Gamma) := \frac{\|\mathcal{P}_i - \mathcal{P}_j\|_p}{\Gamma(h_j)}\left(\frac{h_j}{h_{i,j}}\right)^\kappa$,where $\Gamma \in \Phi^\kappa, \Gamma \not\equiv 0$ and $\mathbb{S} \in \sum_\kappa$.

$\mathfrak{b}_\kappa(\mathbb{s},\Gamma,B) := max_{1\leq i,j\leq n}\{\mathfrak{b}_{i,j}(\mathbb{s},\Gamma) | I_i \subset B$ and $I_j \subset B\}$,

Where an interval $B \subseteq [-1,1]$ contains at least one interval I_v

$\mathfrak{b}_\kappa(\mathbb{s},\Gamma) := \mathfrak{b}(\mathbb{s},\Gamma,I) = max_{1\leq i,j\leq n}\mathfrak{b}_{i,j}(\mathbb{s},\Gamma)$

$c(p) :=$ is an absolute constant depending on p,and is different from one step to others.

$c(\kappa,p) :=$positive constant that are either may only depend on the parameters k and p or absolute .

2 The Auxiliary Lemma

Lemma 2.1 [16]

($n^{-1}\vartheta(x) < \Omega_n(x) < h_j < 5\Omega_n(x), x \in I_j$), ($h_{j\pm1} < 3h_j$).

Lemma 2.2 [13]

Using the same lines word by word used for (6.12), P.235 in[13], we get the following lemma for $0 < p < 1$, ($\mathfrak{g} \in L_p[-1,1], \kappa \in N, a \in [-1,1]$ and $h > 0$ such that $a + (\kappa - 1)h \in [-1,1]$.

$\|\mathfrak{g}(x)\|_p \leq c(p)\left(1 + \frac{|x-a|}{h}\right)^\kappa (w_\kappa(\mathfrak{g},h)_p + \|\mathfrak{g}\|_{[a,a+(\kappa-1)h]})\ x \in [-1,1]$.

Lemma 2.3 [15]

$(|B_{\upsilon}| \leq c\Gamma(h_j)(\frac{h_{i,j}}{h_j})^{\kappa}, x \in I_i, \upsilon = 1,2,3)$.

Lemma 2.4 [14]

$(\Omega_n^2(x) < 4\Omega_n(x_j)(|x - x_j| + \Omega_n(x_j)) < 8h_j(|x - x_j| + \Omega_n(x))$.

Lemma 2.5 [14]

$(\Omega_n(x) + dist(x, I_j) \leq \Omega_n(x) + |x - x_j| \leq 16(\Omega_n(x) + dist(x, I_j))$.

Lemma 2.6 [14]

$(\sum_{j=1}^{n} \widetilde{T}_{j,n_1}(x) \equiv 1, x \in [-1,1])$.

Lemma 2.7 [14]

$$(\left|\widetilde{T}_{j,n_1}^{(q)}(x)\right| \leq c\frac{\delta_n^{S_2}(x)}{\Omega_{n_1}^{q}(x)}\left(\frac{\Omega_{n_1}(x)}{\Omega_{n_1}(x)+diist(x,I_j)}\right)^{\mathcal{G}_2} ,0 \leq q \leq S_2).$$

Lemma 2.8 [14]

$(b_{i,j}(\mathbb{s}, \Gamma) := \frac{\|\mathcal{P}_i - \mathcal{P}_j\|_{I_i}}{\Gamma(h_j)}(\frac{h_j}{h_{i,j}})^{\kappa} ,1 \leq i, j \leq n)$.

Lemma 2.9 [14]

$(\sum_{j=1}^{n}(\frac{\Omega_n(x)}{\Omega_n(x)+dist(x,x_j)})^4 \leq c)$.

Lemma 2.10 [14]

$(|\sigma_1(x)| \leq c b_k(\mathbb{s}, \Gamma, B)\delta^{\gamma}\frac{\Gamma(\Omega)}{\Omega^{q}}$,

$$|\sigma_2(x)| \leq c b_k(\mathbb{s}, \Gamma)\delta^{\gamma}\frac{\Gamma(\Omega)}{\Omega^{q}}\frac{n}{n_1}(\frac{\Omega}{\Omega + dist(x, [-1,1])\backslash B})^{\gamma+1}$$

$|\sigma_3(x)| \leq c b_{\kappa}(\mathbb{s}, \Gamma)\delta^{\gamma}\frac{\Gamma(x)}{\Omega^{q}}\frac{n}{n_1}(\frac{\Omega}{dist(x,[-1,1])\backslash B})^{\gamma+1})$.

Lemma 2.11 [16]

$(1\text{-}x_{j-1} < \int_{-1}^{1}\mathcal{F}_j(t)dt < 1 - x_j, 1 \leq j \leq n)$.

3 Properties of Piecewise Polynomials

Proposition(3.1)

$\mathcal{L}$et $\Gamma \in \Phi^{\kappa}, \kappa \in N, \mathcal{F} \in L_p(I)$and $\mathbb{S} \in \sum_{\kappa,n}$, If $w_{\kappa}(\mathcal{F}, t)_p \leq c(p)\Gamma(t)$and $\|\mathcal{F} - \mathbb{S}\|_p \leq c(p)\Gamma(\Omega_n(x))$ then

$$b_{\kappa}(\mathbb{s}, \Gamma) \leq c(\kappa, p)$$

Proof.

Recalled that $\Gamma \not\equiv 0$,so that $\Gamma(x) > 0, x > 0$. For $1 \leq i, j \leq n$,we have

$$b_{i,j}(\mathbb{s}, \Gamma) \leq \frac{\|\mathcal{P}_i - \mathcal{F}\|_p}{\Gamma(h_j)}\left(\frac{h_j}{h_{ij}}\right)^{\kappa} + \frac{\|\mathcal{F} - \mathcal{P}_j\|_p}{\Gamma(h_j)}\left(\frac{h_j}{h_{i,j}}\right)^{\kappa} := \sigma_1 + \sigma_2 ,$$

where $\|\mathcal{P}_i - \mathcal{F}\|_p = (\int_{-1}^{1}(\mathcal{P}_i - \mathcal{F})^p)^{\frac{1}{p}}$

$$\|\mathcal{F} - \mathcal{P}_j\|_p = (\int_{-1}^{1}(\mathcal{F} - \mathcal{P}_j)^p)^{\frac{1}{p}}$$

we notice that now, for any $1 \leq v \leq n$,inequality

$\|\mathcal{F}(x) - \mathbb{S}(x)\|_p \leq c(p)\Gamma(\Omega_n(x)), x \in [-1,1]$,and by(Auxiliary lemma (2.1)).

$$\frac{\vartheta(x)}{n} < \Omega_n(x) < h_j < 5\Omega_n(x), x \in I_j, 1 \leq j \leq n , h_{j\mp1} < 3h_j ,1 \leq j \leq n$$

$\|\mathcal{P}_{\upsilon} - \mathcal{F}\|_{p(I_{\upsilon})} \leq c(p(I_{\upsilon}))\|\Gamma(\Omega_n)\|_{p(I_{\upsilon})} \leq c(p(I_{\upsilon}))\Gamma(h_{\upsilon})$

Hence , $\sigma_1 \le 1$, where we used the fact that if $h_i \le h_j$,

then $\Gamma(h_i) \le \Gamma(h_j)$, and if $h_i > h_j$ then $\Gamma(h_i)/\Gamma(h_j) \le {h_i}^{k} / {h_j}^{k}$.

To estimate σ_2,let us remember the following fact: by using (Auxiliary lemma(2.2)) ($\|g(x)\|_p \le c(p)\left(1+\frac{|x-a|}{h}\right)^{\kappa}(w_\kappa(g,h)_p + \|g\|_{[a,a+(\kappa-1)h]})$ $x \in [-1,1]$).

setting $a := x_j$ $g := \mathcal{F} - \mathcal{P}_j$, and $h := \frac{h_j}{max\{1,\kappa-1\}}$,and note that

$w_\kappa(g,h)_p = w_\kappa(\mathcal{F}-\mathcal{P}_j,h)_p = w_\kappa(\mathcal{F},h)_p \le c(p)\Gamma(h)$, we get

$\|\mathcal{F}(x) - \mathcal{P}_j(x)\|_p \le c(p)(1+\frac{|x-x_j|}{h_j})^k\left(\Gamma(h_j) + \|\mathcal{F}-\mathcal{P}_j\|_p\right), x \in I$,where $I = [-1,1]$.

and so, $\|\mathcal{F}-\mathcal{P}_j\|_p \le c(p)(\frac{h_{i,j}}{h_j})^k\Gamma(h_j)$. therefore, $\sigma_2 \le c$.

Proposition(3.2)

$\mathcal{L}$et $\kappa \ge 3, \Gamma \in \Phi^\kappa$and $\mathbb{S} \in \sum^{1}_{\kappa;\mathfrak{n}}$ Then $\mathfrak{b}_\kappa(\mathbb{S}) \le c(p)\left\|\frac{\Omega^2 \mathbb{s}''}{\Gamma(\Omega)}\right\|_p$,

Proof.

since $\mathcal{P}_j(x) = \mathbb{s}(-1) + \mathbb{s}'(-1)(x+1) + \int_{-1}^{x_j}(x-u)s''(u)du + \int_{x_j}^{x}(x-u)\mathcal{P}''_j(u)du.$ and

$\mathcal{P}_i(x) = \mathbb{s}(-1) + \mathbb{s}'(-1)(x+1) + \int_{-1}^{x_i}(x-u)s''(u)du + \int_{x_i}^{x}(x-u)\mathcal{P}''_i(u)du.$

We have $\mathcal{P}_j - \mathcal{P}_i = \int_{x_i}^{x_j}(x-u)s''(u)du + \int_{x_j}^{x}(x-u)p''_j(u)du - \int_{x_i}^{x}(x-u)p''_i(u)du. =: B_1(x) + B_2(x) + B_3(x)$,then

$\|\mathcal{P}_j - \mathcal{P}_i\|_p \le c(p)(\|B_1(x)\|_p + \|B_2(x)\|_p + \|B_3(x)\|_p)$.

So by (Auxiliary lemma(2.3)) ($|B_v| \le c\Gamma(h_j)(\frac{h_{i,j}}{h_j})^\kappa, x \in I_i, v = 1,2,3$

$\|B_3(x)\|_p \le c(p)\Gamma(h_j)(\frac{h_{i,j}}{h_j})^\kappa$, $\|B_2(x)\|_p \le c(P)\Gamma(h_j)\frac{(h_{i,j})^{\kappa-1}}{{h_j}^{\kappa}}$.

$\|B_1(x)\|_p \le h_{i,j}B_1$, where $B_1 \le c(p)\Gamma(h_j)\frac{(h_{i,j})^{k-1}}{h_j^k}$

So $max_{1\le i,j\le n}\{\mathfrak{b}_{i,j}(\mathbb{s},\Gamma)\} = \mathfrak{b}_\kappa(s,\Gamma)$

So $\frac{\|\mathcal{P}_j-\mathcal{P}_i\|_p}{\Gamma(h_j)}(\frac{h_i}{h_{i,j}})^\kappa =: \mathfrak{b}_{i,j}(s,\Gamma)$,we have $\mathfrak{b}_\kappa(\mathbb{s}) \le \|B_v(x)\|_p \le c(p)\Gamma(h_j)(\frac{h_{i,j}}{h_j})^\kappa, x \in I_i, v = 1,2,3.$

(i. e$\|\mathcal{P}_j - \mathcal{P}_i\|_p \le c(p)\Gamma(h_j)(\frac{h_{i,j}}{h_j})^k$. This prove is complete

theorem(3.3)

$\mathcal{L}$et $\kappa \in N, \gamma > 0, \Gamma \in \Phi^\kappa$,and let $\mathfrak{n}, \mathfrak{n}_1 \in N$ be such that $\frac{\mathfrak{n}_1}{\mathfrak{n}}$

If $\mathbb{S} \in \sum_{\kappa,\mathfrak{n}}$,Then there exist a polynomial $\mathcal{D}_{\mathfrak{n}1}(.,\mathbb{s})$ of degree$\le c\mathfrak{n}_1$ .such that

$\|\mathbb{S}(x) - \mathcal{D}_{\mathfrak{n}1}(x,\mathbb{s})\|_p \le c(p)\delta_\mathfrak{n}^\gamma(x)\Gamma(\Omega_\mathfrak{n}(x))\mathfrak{b}_k(\mathbb{s},\Gamma)$ (3.4)

Overtime, if $\mathbb{S} \in {L_p}^{\mathfrak{r}-1}(I), \mathfrak{r} \le \kappa$, for some $\mathfrak{r} \in N$ and

$B := [X_{M^*}, X_{M_*}], 0 \le M_* \le M^* \le n$,then for all $x \in B\backslash\{x_j\}_{j=1}^{n-1}$ and $0 \le q \le \mathfrak{r}$, we have

$\left\|\mathbb{S}^{(q)}(x) - \mathcal{D}_{\mathfrak{n}1}{}^{(q)}\right\|_p \le c(p)\delta_n^\gamma(x)\frac{\Gamma(\Omega_\mathfrak{n}(x))}{\Omega_\mathfrak{n}^q(x)}(\mathfrak{b}_\kappa(\mathbb{s},\Gamma,B) + \mathfrak{b}_\kappa(\mathbb{s},\Gamma)\ \mathfrak{n}/\mathfrak{n}_1\left\|(\frac{\Omega_\mathfrak{n}(x)}{dist(x,[-1,1]\backslash B})^{\gamma+1}\right\|_p$ (3.5)

The constants $c(p)$ is depending on p only and not on $\mathfrak{n}$.

Proof.

Let $\mathcal{D}_{\mathfrak{n}1} := \sum_{j=1}^{\mathfrak{n}}\mathcal{P}_j(x)\,\widetilde{T}_{j,\mathfrak{n}1}(x)$ (3.6)

where $\widetilde{T}_{j,n1}$denoted the polynomials of degree$\le c(\mathcal{S}_2,\mathcal{G}_2)$ from the statement of [14.corollary 7.2].and $\mathcal{D}_{\mathfrak{n}1}(.,\mathbb{s})$denoted the polynomial of degree$< \kappa + c(\mathcal{S}_2,\mathcal{G}_2)$.The parameters $\mathcal{S}_2$and $\mathcal{G}_2$ are chosen to be sufficiently large and depend on γ and κ.

So let $\mathcal{S} = \gamma$ and $\mathcal{G}_2 = \mathcal{S} + 4\kappa + 5\mathcal{S}$ .so by(Auxiliary Lemma(2.4) and(2.5))

$(\Omega_n^2(x) < 4\Omega_n(x_j)(|x - x_j| + \Omega_n(x_j)) < 8h_j(|x - x_j| + \Omega_n(x))$

$(\Omega_n(x) + dist(x, I_j) \le \Omega_n(x) + |x - x_j| \le 16\left(\Omega_n(x) + dist(x, I_j)\right))$.

Imply

$\frac{h_v}{h_j} < 5\frac{\Omega}{h_j} < 40\frac{|x-x_j|+\Omega}{\Omega} \sim \frac{\Omega+dist(x,I_j)}{\Omega}$,$1 \le j \le n$, $1 \le v \le n$ (3.7)

Also

$\frac{h_{v,j}}{h_v} \le c(p)\ \frac{\Omega+dist(x,I_j)}{\Omega}$,$1 \le j \le n$ (3.8)

Indeed, if $|j - v| \le 1$,then by (Auxiliary Lemma (2.1))

Implies that $h_{v,j} \sim h_v$.

If$|j - v| \ge 2$, since we have unique $I_i$between $I_v$and I_j and then using by (Auxiliary Lemma(2.1))]to get:

$h_{v,j} = h_v + h_j + dist(I_v, I_j) \le h_v + 4dist(I_v, I_j)$

$\le h_v + dist(x, I_j)$, and by (3.5)follows .

Since $\mathbb{S}(x) = p_v(x)$,(Auxiliary lemma(2.6)) $(\sum_{j=1}^{n} \tilde{T}_{j,n_1}(x) \equiv 1)$ implies $\mathbb{S}(x) - \mathcal{D}_{n1}(x, \mathbb{s}) = \mathbb{S}(x)\sum_{j=1}^{n} \tilde{T}_j(x) - \sum_{1\le j\le n, j\ne v} \mathcal{P}_j(x)\tilde{T}_j(x)$, And so

$$\mathbb{S}^{(q)}(x) - \mathcal{D}_{n1}{}^{(q)}(x, \mathbb{s}) = \sum_{1\le j\le n, j\ne v} \left(\left(\mathcal{P}_v(x) - \mathcal{P}_j(x)\right)\tilde{T}_j(x)\right)^{(q)}$$

$$= \sum_{1\le j\le n, j\ne v} \sum_{i=0}^{q} \binom{q}{i} \left(\mathcal{P}_v{}^{(i)}(x) - \mathcal{P}_j{}^{(i)}(x)\right)\tilde{T}_j{}^{(q-i)}(x),$$

With the assumption that $x \notin \{x_j\}_{j=1}^{n-1}$, if$q \ge 1$, since $\mathbb{S}^{(q)}$that is not exist at those point. note also that $x \in D_j$ for all $1 \le j \le n, j \ne v$,and(Auxiliary Lemma(2.7))

$\left\|\tilde{T}_{j,n_1}^{(q)}(x)\right\|_p \le c(p)\frac{\delta_n^{\mathcal{S}_2}(x)}{\Omega_{n_1}^{q}(x)}\left(\frac{\Omega_{n1}(x)}{\Omega_{n1}(x)+dist(x,I_j)}\right)^{\mathcal{G}_2}$, $0 \le q \le \mathcal{S}_2$ can be used

for any $\tilde{T}_j$ above. since $\Gamma(h_j) \le \Gamma(h_{v,j}) \le \Gamma(h_v)(\frac{h_{v,j}}{h_v})^k \le c(p)\Gamma(\Omega)(\frac{h_{v,j}}{h_v})^\kappa$,

it follow from [14],Auxiliary Lemma (2.8)] $\mathfrak{b}_{i,j}(\mathbb{s}, \Gamma) = \frac{\|\mathcal{P}_i - \mathcal{P}_j\|_p}{\Gamma(h_j)}(\frac{h_j}{h_{i,j}})^{\kappa k}$ and [Lemma(3.1)],(3.7)and(3.8) for all $i \ge 0$.

$\left\|\mathcal{P}_v{}^{(i)} - \mathcal{P}_j{}^{(i)}\right\|_{p(I_v)} \le c(p)h_v^{-i}\left\|\mathcal{P}_v - \mathcal{P}_j\right\|_{p(I_v)}$

$\le c(p)\mathfrak{b}_{v,j}(\mathbb{s}, \Gamma)\frac{\Gamma(h_j)}{h_v^i}(\frac{h_{v,j}}{h_j})^\kappa$ (3.9)

$$\le c(p)\ \mathfrak{b}_{v,j}(\mathbb{s}, \Gamma)\frac{\Gamma(\Omega)}{\Omega^i}(\frac{h_{v,j}{}^2}{h_v h_j})^\kappa$$

$$\le c(p)\mathfrak{b}_{v,j}(\mathbb{s}, \Gamma)\frac{\Gamma(\Omega)}{\Omega^i}(\frac{\Omega + dist(x, I_j))}{\Omega}^{3\kappa}.$$

observing that $\frac{\Omega_1}{\Omega_1+dist(x,I_j)} \le \frac{\Omega}{\Omega+dist(x,I_j)}$ (3.10)

and using (Auxiliary Lemma(2.7)).we now conclude that ,for all

$0 \le i \le q$ and $1 \le j \le n, j \ne v$, $\left\|(\mathcal{P}_v{}^{(i)}(x) - \mathcal{P}_j{}^{(i)}(x))\tilde{T}_j{}^{(q-i)}(x)\right\|_p$

$\le c(p)\mathfrak{b}_{v,j}(\mathbb{s}, \Gamma)\delta^{\mathcal{S}_2}\frac{\Gamma(\Omega)}{\Omega^i\Omega_1{}^{q-i}}(\frac{\Omega_1}{\Omega_1+dist(x,I_j)})^{\mathcal{G}_2-3k}$, If $i = q$, this become

$\left\|\left(\mathcal{P}_v{}^{(q)}(x) - \mathcal{P}_j{}^{(q)}(x)\right)\tilde{T}_j(x)\right\|_p \le c(p)\ \mathfrak{b}_{v,j}(\mathbb{s}, \Gamma)\delta^{\mathcal{S}_2}\frac{\Gamma(\Omega)}{\Omega^q}(\frac{\Omega_1}{\Omega_1+dist(x,I_j)})^{\mathcal{G}_2-3\kappa}$ (3.11)

and ,in particular , if $i = q = 0$,

$$\left\|\left(\mathcal{P}_v(x) - \mathcal{P}_j(x)\right)\tilde{T}_j(x)\right\|_p \le c(p)\mathfrak{b}_{v,j}(\mathbb{s}, \Gamma)\delta^{\mathcal{S}_2}\Gamma(\Omega)(\frac{\Omega_1}{\Omega_1+dist(x,I_j)})^{\mathcal{G}_2-3\kappa} \quad (3.12)$$

If we assume $j \neq s \pm 1$,we get $dist(x, I_j) > \Omega/3$ therefore $\frac{\Omega_1}{\Omega} \leq \frac{n}{n_1}$, implies

$$\left\|\left(\mathcal{P}_v^{(i)}(x) - \mathcal{P}_j^{(i)}(x)\right)\tilde{T}_j^{(q-i)}(x)\right\|_p \leq c(p)b_{v,j}(s,\Gamma)\delta^{S_2}\frac{\Gamma(\Omega)}{\Omega^q}\frac{\Omega_1}{\Omega}\left(\frac{\Omega}{\Omega_1 + dist(x, I_j)}\right)^{q-i+1}\left(\frac{\Omega_1}{\Omega_1 + dist(x, I_j)}\right)^{G_2-3\kappa-q+i-1}$$

$$\leq c(p)b_{v,j}(s,\Gamma)\delta^{S_2}\frac{\Gamma(\Omega)}{\Omega^q}\frac{n}{n_1}\left(\frac{\Omega_1}{\Omega_1+dist(x, I_j)}\right)^{G_2-3\kappa-q-1} \qquad (3.13)$$

Now let us study the case $q \geq 1$, $i \leq q - 1$ and $j = v \pm 1$,

we study the case $j = v + 1$,the case $j = v - 1$ of study similar to above. Since $\mathbb{S}$ is smooth. In fact

If $\mathbb{S} \in L_p^{q-1}[-1,1]$,we have $\mathcal{P}_v^{(\iota)}(x_v) = \mathcal{P}_{v+1}^{(\iota)}(x_v)$, $0 \leq \iota \leq q - 1$ and so by (3.4) ,(3.10).

$$\left\|\mathcal{P}_v^{(i)}(x) - \mathcal{P}_{v+1}^{(i)}(x)\right\|_p = \left\|\frac{1}{(q-i-1)!}\int_{x_v}^{x}(x-u)^{q-i-1}(\mathcal{P}^{(q)}(u) - \mathcal{P}_{v+1}^{(q)}(u)du\right\|_p$$

$$\leq \frac{1}{(q-i-1)!}\left(\sum_{i=1}^{n}|x_i - u|^q\right)^{\frac{1}{q}}\left(\sum_{i=1}^{n}|p_v(x_i) - p_{v+1}(x_i)|^p\right)^{\frac{1}{p}}$$

where $\mathcal{P}_v, \mathcal{P}_{v+1}, x - u$ are polynomials so

$\leq \frac{c(p)}{(q-i-1)!}\|x-u\|_{p(I_v)}\left\|\mathcal{P}_v^{(q)}(u) - \mathcal{P}_{v+1}^{(q)}(u)\right\|_{p(I_v)}$, where $0 < p < 1$.

$\leq \frac{c(p)}{(q-i-1)!}\|x-u\|_{p(I_v)}b_{v,v+1}(s,\Gamma)\frac{\Gamma(\Omega)}{\Omega^q}\left(\frac{\Omega+|x-x_v|}{\Omega}\right)^{3\kappa}$,

Therefore , $\left\|\left(\mathcal{P}_v^{(i)}(x) - \mathcal{P}_v^{(i)}(x)\right)\tilde{T}_{v+1}^{(q-i)}(x)\right\|_p \leq c(p)b_{v,v+1}(s,\Gamma)\delta^{S_2}\frac{\Gamma(\Omega)}{\Omega^q\Omega^{q-i}}\left\|(x-u)_{p(I_v)}\left(\frac{\Omega_1}{\Omega_1+|x-x_v|}\right)^{G_2-3\kappa}\right\|_{p(I_v)}$

In summary ,the estimate $\left\|\left(\mathcal{P}_v^{(i)}(x) - \mathcal{P}_{v\pm1}^{(i)}(x)\right)\tilde{T}_{v\pm1}^{(q-i)}(x)\right\|_p \leq$

$$c(p)b_{v,v\pm1}\delta^{S_2}\frac{\Gamma(\Omega)}{\Omega^q}\left\|\left(\frac{\Omega_1}{\Omega_1+dist(x, I_{v\pm1})}\right)^{G_2-3\kappa-q}\right\|_p \qquad (3.14)$$

is valid for all $0 \leq i \leq q$ provided that $\mathbb{S} \in L_p^{q-1}[-1,1]$

(for $i = q$ it follows from (3.4) (3.13),(3.12),(3.11)].and using

(Auxiliary Lemma(2.9))and the estimate $b_{v,j}(s,\Gamma)$, we have

$$\|\mathbb{S}(x) - \mathcal{D}_{n1}(x,s)\|_p \leq c(p)b_\kappa(s,\Gamma)\delta^{S_2}\Gamma(\Omega)\left\|\sum_{1\leq j\leq n, j\neq v}\left(\frac{\Omega}{\Omega+dist(x, I_j)}\right)^{G_2-3\kappa}\right\|_p \qquad (3.15)$$

$\leq c(p)b_k(s,\Gamma)\delta^\gamma\Gamma(\Omega)$, and (3.5)is proved .

Let us now estimate (3.6). Assume $\mathbb{S} \in L_p^{r-1}[-1,1]$and $o \leq q \leq r$

we write $\mathbb{S}^{(q)}(x) - \mathcal{D}_{n1}^{(q)}(x,s) = \sum_{1\leq j\leq n, j\neq v}\left(\left(\mathcal{P}_v(x) - \mathcal{P}_j(x)\right)\tilde{T}_j(x)\right)^{(q)}$

$$=: \left(\sum_{j\in Z_1} + \sum_{j\in Z_2} + \sum_{j\in Z_3} + \sum_{j\in Z_4}\right)\left((\mathcal{P}_v(x) - \mathcal{P}_j(x))\tilde{T}_j(x)\right)^{(q)}$$

$$\left\|S^{(q)}(x) - \mathcal{D}_{n1}^{(q)}(x,s)\right\|_p =: \sigma_1(x) + \sigma_2(x) + \sigma_3(x) + \sigma_4(x),$$

Where

$Z_1 = \{j | 1 \leq j \leq n , I_j \subset B , j \neq v, v \pm 1\}$

$Z_2 = \{j | 1 \leq j \leq n, I_j \not\subset B , j \neq v , v \pm 1\}$

$Z_3 = \{j | 1 \leq j \leq n, j = v + 1 \}$

$Z_4 = \{j | 1 \leq j \leq n , j = v - 1 \}$.

so by (Auxiliary Lemma (2.10)), $\|\sigma_1(x)\|_p \leq c(p)b_k(s,\Gamma,B)\delta^\gamma\frac{\Gamma(\Omega)}{\Omega^q}$ also

$$\|\sigma_2(x)\|_p \leq c(p)b_k(s,\Gamma)\delta^\gamma\frac{\Gamma(\Omega)}{\Omega^q}\frac{n}{n_1}\left\|\left(\frac{\Omega}{\Omega + dist(x, [-1,1]\backslash B)}\right)^{\gamma+1}\right\|_p$$

$$\|\sigma_3(x)\|_p \leq c(p)b_k(s,\Gamma)\delta^\gamma\frac{\Gamma(\Omega)}{\Omega^q}\frac{n}{n_1}\left\|\left(\frac{\Omega}{dist(x, [-1,1]\backslash B)}\right)^{\gamma+1}\right\|_p$$

similarly $\|\sigma_4(x)\|_p$ is completely analogous ,so the prove of this theorem is complete .

4 L_p convex approximation by piecewise polynomial

theorem(4.1)

Let $\mathcal{S} > 0$, $\kappa \in N$ and $\Gamma \in \Phi^{\kappa}$, be given. If $\mathbb{S} \in \sum_{\kappa,n} \cap \Delta^{(2)}$,is such that $\|\mathbb{S}''(x)\|_p \le c(p)\frac{\Gamma(\Omega_n(x))}{\Omega_n^2(x)}, x \in [x_{n-1}, x_1] \setminus \{x_j\}_{j=1}^{n-1}$ (4.2)

$0 \le \mathbb{S}'(x_j +) - \mathbb{S}'(x_j -) \le \frac{\Gamma(\Omega_n(x_j))}{\Omega_n(x_j)}, 1 \le j \le n-1$ (4.3) and

$\mathbb{S}''(x) = 0 , x \in [-1, x_{n-1}) \cup (x_1, 1]$ (4.4)

Then there is a polynomial $\mathcal{P} \in \Delta^{(2)} \cap \pi_{cn}$, $c = c(p,\kappa,\mathcal{S})$,such that$\|\mathbb{S}(x) - \mathcal{P}(x)\|_p \le c(p,\kappa,\mathcal{S})\delta_n^{\mathcal{S}}(x)\Gamma(\Omega_n(x)), x \in [-1,1]$ (4.5)

Proof.

Let $\mathbb{S}_1$is denote continuous piecewise linear function interpolates $\mathbb{S}$ at the points $x_j, o \le j \le n$,and $\iota_j = \mathbb{S}_1/I_j$ then

$\mathbb{S}_1 \in \Delta^{(2)}, \mathbb{S}_1(x) = \mathbb{S}(x), x \in I_1 \cup I_n$ (4.6)

And , for $x \in I_j, 1 \le j \le n$, using Whitney's inequality and (Auxiliary lemma(2.1)) ($n^{-1}\vartheta(x) < \Omega_n(x) < h_i < 5\Omega_n(x)$, $x \in I_j$

$\|\mathbb{S}(x) - \mathbb{S}_1(x)\|_{p(I_j)} \le c(p)\omega_2(\mathbb{s}, h_j, I_j)_p$

$\le c(p)h_j\omega_1(\mathbb{s}', h_j, I_j) \le c(p)h_j^2\|\mathbb{s}''\|_{p(I_j)}$,so by using $\|\mathbb{s}''\|_p \le c(p)\frac{\Gamma(h_j)}{h_j^2}$

We get $\|\mathbb{S}(x) - \mathbb{S}_1(x)\|_{p(I_j)} \le c(p)\Gamma(h_j)$, So we have by proposition

(3.1), $\|\mathbb{S}(x) - \mathbb{S}_1(x)\|_p \le c(p)\Gamma(\Omega_n(x)), x \in [-1,1])$ (4.7)

So we can be write $\mathbb{S}_1$as

$$\mathbb{S}_1(x) = \mathbb{S}_1(-1) + \mathbb{S}'_1(-1)(x+1) + \sum_{j=1}^{n-1} \mathcal{S}_i\Phi_j(x),$$

$\mathcal{S}_j := \mathbb{S}'_1(x_j +) - \mathbb{S}'_1(x_j -)$, note that, by Markov and Whitney

$\|\mathcal{P}'\|_p \le c(p,n)n^2\|p\|_p$.

$$0 \le \mathcal{S}_j = \iota'_j(x_j) - \iota'_{j+1}(x_j) \le c(p)h_j^{-1}\|\iota_j - \iota_{j+1}\|_{p(I_j \cup I_{j+1})}$$

$\le c(p)h_j^{-1}\omega_2(\mathbb{s}, h_j, p(I_j \cup I_{j+1}))_p$

$\le c(p)h_j\left(\|\mathbb{S}''\|_{p(I_j)} + \|\mathbb{S}''\|_{p(I_{j+1})}\right) + c(p)(\mathbb{S}'(x_j +) - \mathbb{S}'(x_j -))$

$\le c(p)h_j^{-1}\Gamma(h_j), 1 \le j \le n-1$.

Now , if $\mathcal{P}(x) := \mathbb{S}_1(-1) + \mathbb{S}'_1(-1)(x+1) + \sum_{j=1}^{n-1}\mathcal{S}_i F_j(x)$,

So that $\mathcal{P}$ is a polynomial of degree not exceeding cn, and also convex. and using[proposition (3.1) (4.6)and(4.7) ,we only need to estimate $\|\mathbb{S}_1(x) - \mathcal{P}(x)\|_p$. Note that the inequality($c\mathcal{T}_j^2(x)\Omega_n(x) \le h_j \le c\mathcal{T}_j^{-1}(x)\Omega_n(x)$ implies, For all $1 \le j \le n$ and $x \in [-1,1]$,

$\Gamma(h_j) \le \Gamma(c\mathcal{T}_j^{-1}(x)\Omega_n(x)) \le c\mathcal{T}_j^{-k}(x)\Gamma(n(x))$.Hence by(Auxiliary lemma(2.11),(2.18)) , We have

$$\|\mathbb{S}_1(x) - \mathcal{P}(x)\|_p \le \sum_{j=1}^{n-1}\mathcal{S}_i|\Phi_j(x) - F_j(x)| \le c(p)\sum_{j=1}^{n-1}\Gamma(h_j)\delta_n^{\mathcal{S}}(x)\mathcal{T}_j^{\mathcal{G}}(x)$$

$\le c(p)\delta_n^{\mathcal{S}}\Gamma(\Omega_n(x))\sum_{j=1}^{n-1}\mathcal{T}_j^{\mathcal{G}-k}(x)$

$\le c(p)\delta_n^{\mathcal{S}}(x)\Gamma(\Omega_n(x))$, when $\mathcal{G} \ge k+2$.

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