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A DOUBLE WEIGHT COPOSITIVE APPROXIMATION

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Abstract.

Many results proved for the direct and inverse approximation theorems in terms of the kth order modulus of smoothness. Sometimes we need that the approximation have the same geometric properties of the target function. Here we introduce a direct theorem for co positive approximation in terms of double weights modulus of smoothness and we see that this shape preserving approximation restricts the order of the modulus of smoothness say one only. In the second result of this paper a strength of the first result it mean the first result is in terms of $\omega_{\varphi}^{1}(f,\delta)$ not $\omega_{\varphi}^{2}(f,\delta)_{p,w}$.

Keywords. modulus of smoothness, polynomial, positive polynomial, positive approximation,

1.Introduction

Many direct theorems for functions in L_p spaces for $0\leq p \leq \infty$, were introduced for the approximation by algebraic, trigonometric, splines, wavelets and neural networks, these theorems in terms of the Kth order ordinary modulus of smoothness see for example ((1) (2) (3) (4) (5) (6) (7)(9)

In(8)

K. A. Kopotun introduced the double weights modulus of smoothness, and he defined it as:

If $r \in \mathbb{N}$, The weighted modulus of smoothness is defined by

$$w_{\varphi}^{r}(f,t)_{p,w} := \sup_{0<h\leqq t} || \triangle_{h\varphi(.)}^{r} (f,.)||_{p,w} \quad \ldots\ldots (1)$$

where $\varphi(x) := \sqrt{1-x^2}$.

The approximation in the papers ((1) (2) (3) (4) (5) (6) (7) (9)) is unconstrained. But we sometimes need that approximation have the same geometric properties of the target function such as positivity monotonicity and convexity. Here we introduce direct theorem for the co positive approximation for functions in L_p spaces for $0 < p < 1$, using algebraic polynomials and in terms of the double weighted modulus of smoothness. We see that the positivity constrain restricts the order of the modulus of smoothness to 1, that we prove in the first result and we strength it in the second theorem.

approximation.

$y \in Y$ is best approximation to $x \in X$ *if*

$$||x-y|| := \inf_{g\in Y} ||x-g||, \qquad (2)$$

Co positive**.**

the two functions f and g are called co positive if f. g > 0.

The degree of co positive approximation by algebraic polynomial by algebraic polynomial of degree $\leq n$ is of functions in $L_{P,W}(1)$ is,

$$E_n^0(f)_{p,w} := \inf_{p_n \in P_n \cap L_{p,w}(1) \cap \Delta^0} ||f-p_n||,$$

Where Δ^0 is the set of all piecewise positive functions.

$(L_{p,w})$ space.[9]

the space of all function that is $\|f\|_{p,w} < \infty$, define

$$\|f\|_{Lp,w}(I) = \left(\int_I |f(u)|^p\, w(u)du\right)^{\frac{1}{p}}, \qquad 0 < P < 1 \qquad (3)$$

Doubling constant of the weight w.[9]

let w be the doubling weight if there is a counstant L such that

$$\int_{2I} w(u)du \leqq L \int_I w(u)du. \qquad (4)$$

S.N. Bernstein inequality. In mathematical analysis, Bernstein's inequality states that on the complex plane, within the disk of radius 1, the degree of a polynomial times the maximum value of a polynomial is an upper bound for the similar maximum of its derivative. Taking the k-th derivative of the theorem,

$$\max_{|z|\leq 1}(|p^k(z)|) \leq \frac{n!}{(n-k)!}\cdot \max_{(z)\leq 1}(|p^k(z)|) \qquad (5)$$

2.Auxiliary Results

In this section we introduce the results that we need for the proof of our main theorem.

Lemma 2.1.[1]

For odd monotone increasing q_n , we have

$$|sgn\,(x) - q_n(x)| \leq C_3|nx|^{-3}, \quad x \in [-1,0) \cup (0,1], \qquad (1)$$

$$|sgn(x) - q_n(x)| \leq 1, \quad -1 \leq x \leq 1.$$

Lemma 2.2.

There exist a constant C(p) with the following property for f$\in L_p[-1,1]$be a piecewise monotone function with $d \geq 1$ changes of monotonicity in all , one of which is at zero where f has a zero . define

$$\hat{f}(x) = \begin{cases} f(x) & ,x \geq 0 \\ -f(x) & ,x < 0 \end{cases} .$$

And suppose that for same n $\geq$ 1 and some $\in\geq \omega_\phi(\hat{f}' ,\frac{1}{n})$p,w there exists a polynomial $p_n \in \Pi_n$ comonotone with $\hat{f}$ *such that*

$$\|\hat{f} - p_n\|_{p,w} \leq \frac{\varepsilon}{n}, \qquad (2)$$

$$\|\hat{f}' - p'_n\|_{p,w} \leq \varepsilon$$

Then there exists a polynomial $p_{2n} \in \Pi_{2n}$. Comonotone with f satisfying

$$||f - p_{2n}||_{p,\omega} \leq C\,\frac{\varepsilon}{n}, \qquad ||\hat{f}' - p'_{2n}||_{p,w} \leq C\varepsilon \qquad (3)$$

Proof.

$\hat{f}$ has one less change of monotonicity than f . Also for 0 <|x|< $\frac{k}{n}$, $k\geq 1$.

$$|\hat{f}'(x)| \leq C(k)\Delta'_{\frac{1}{n}}(\hat{f}'). \qquad (4)$$

Since $\hat{f}'(0) = 0$ *and since* $\hat{f}(0) = 0$.

$$\|\hat{f}'(x)\|_{p,w} \leq \frac{k^2}{n} w_\varphi(\hat{f},\tfrac{1}{n})_{p,w}, \qquad (5)$$

we follow Devore $[5, pp. 908 - 909]$in constructing for each $n \geq 1$ the approximation to $sgn(x)$,

$$q_n(x) = C\int_0^x (\tfrac{T_m(x)}{t})^4\; dt,$$

where m is the largest odd integer so that $q_n \in \Pi_n$ and C is chosen so that $q_n(1) = 1$. q_n is odd ,monotone increasing then using lemma 2.1 ,we have

$$|sgn(x) - q_n(x)| \leq C|nx|^{-3}\;, \; x \in [-1,0) \cup (0,1] \qquad (6)$$

$$|sgn(x) - q_n(x)| \leq 1 \quad ,-1 \leq x \leq 1$$

Since$\hat{f}(0) = 0$ we may assume $p_n(0) = 0$ replacing ε by 2ε in the first inequality of (1).Define

$$p_{2n}(x) = \int_0^x p'_n(x)\, q_n(x)dt \,. then\; p_{2n}\; is\; comonotone\; with\; f\;. also$$

$$f(x) - p_{2n}(x) = \int_0^x [\hat{f}'(t) - p'_n(t)\,dt]\; sgn(t)dt + \int_0^x p'_n(t)[\,sgn(t) - q_n(t)]dt$$

$let\; \eta = \frac{sgn(x)}{n}$, $If\; 0 < |x| \leq \frac{i}{n}, i \geq 1$, then using (2), (4), (6)and that

$\hat{f}'(0) = 0$,we have

$$||\int_0^x p'_n(t)[sgn(t)-q_n(t)dt]dt||_{p,w} \le \sum_{k=0}^{i-1} |\int_{k\eta}^{(k+1)\eta} p'_n(t)[\,sgn\,(t)-\,q_n(t)]$$

from (1) $||\int_0^x p'_n[sgn(t)-q_n(t)dt\}||_{p,w} \le ||\int_0^x \hat{f}'[sgn(t)-q_n(t)dt||_{p,w}$

$$\le ||\int_0^x \Delta|n^{-3}x|dt||_{p,w}$$

$$\le \frac{1}{n}\left[\omega_\varphi\left(\hat{f}',\tfrac{1}{n}\right)_{p,w}+\varepsilon\right]+\frac{1}{n}\sum_{k=1}^{i-1}[(k+1)\omega_\varphi\left(\hat{f}',\tfrac{1}{n}\right)_{p,w}+C]\ C_3k^{-3}$$

$$\le \frac{C_4\varepsilon}{n}.$$

Lemma 2.3.

Let $f \in L_P[a,b]$ be piecewise monotone (change monotonicity r time, $0 < r < \infty)\ in\ [-1,1]$, then for each $n \ge 1$ there is a polynomial p_n co monotone with f such that .

$$\left\|f^{(i)}(x)-\ p^{(i)}{}_n\right\|_{p,w} \le C(r)n^{i-1}\,\omega_\varphi\left(f',\tfrac{1}{n}\right),\ i=0,1 \qquad (7)$$

C(r) is an absolute constant independent of f and n , and independent on r

Proof.

For small n , say n < n(p) the theorem is trivial since if

$$f'(\alpha)=0, since\ |f(x)-f(\alpha)| \le f'(\xi)|x-\alpha|.$$

So $\|f(x)-f(\alpha)\|_{p,w} \le ||(x-\alpha)f'||_{p,w} \le C\ \omega_\varphi(f,2)_{p,w}\ \le C\,\frac{1}{n}\omega_\varphi(f,\frac{1}{n})_{p,w}$

We prove the theorem for large n by induction on s the number of changes of monotonicity. We consider the proposition that there exist constant C(s) and N(s),such that for any piecewise monotone function $f \in L_P[-1,1]\ with\ S \ge 0$ changes of monotonicity, such that f' has zeros in $[-1,1]$ (this assumption is needed only when S=0) , and any $n \ge N(s)$, there exists $a\ p_n \in \Pi_n$ comonotone with f satisfying

$$if\ i=0 \qquad ||f-p_n||_{p,w} \le c(p)\tfrac{1}{n}\,\omega_\varphi\,(f',\tfrac{1}{n})_{p,w} \qquad (9)$$

$$if\ i=1 \qquad \left\|f'-p'_n\right\|_{p,w} \le c(p)\omega_\varphi(f',\tfrac{1}{n})_{p,w}$$

The proposition is true for S=0 as can be seen by using the following construction of Devore [3, p.341] Assume without loss of generality that f vanishes at one of the zeros of $f'(subtract\ a\ constant\ from\ f\ and\ add\ it$ to the approximation if necessary) ,and extend f by linear functions to $[-3,3]$

Preserving the modulus of continuity of f'. Note that

$$max\{\ \|f\|_{[-3,3]}\,,\|f'\|_{[-3,3]}\} \le C_5\ n\ \omega_\varphi\left(f',\tfrac{1}{n}\right)_{p,w} \text{ and define}$$

$$\Lambda_n(f,x)=\int_{-2}^{2}\lambda_n(x-t)f(t)dt+a_nx.$$

where $\{\lambda_n\}$is a suituble sequence of positive polynomial kernels (see Devore [3]) with. $\int_{-4}^{4}\lambda_n(t)dt=1\,,\int_{-4}^{4}\lambda_n(t)t^2dt=O(n^{-2})\ and\ \|\lambda_n\|_{Lp,w}\ =O(n^{-2})$, then

$$\Lambda_n{}'(f,x)=\int_{-2}^{2}\lambda_n(x-t)f'(t)dt+\lambda_n(x+2)f(-2)-\lambda_n(x-2)f(2)+a_n$$

where

$$||\lambda_n(x+2)f(-2)-\lambda(x-2)f(2)||_{p,w} \le C_6n^{-1}\omega_\varphi\left(f',\tfrac{1}{n}\right)_{p,w}\ ,-1 \le x \le 1.$$

It is clear that $\Lambda_n\ (f)$will have the same constant monotonicity as f , and the desired approximation properties, provided an is properly chosen from $\pm C_6n^{-1}\omega_\varphi(f',\frac{1}{n})_{p,w}$.

Assuming now that the properties is true for s-1 , we will prove it for s. Given f with s≥ 1 changes f monotonicity extended its definition to $[-3,3]as$ before , since f has at least one turning point $f'(\alpha)=0$ for some $-1<\alpha<1$ working with the interval, I, of length 4 centered at α (obviously $I \subset [-3,3]$)we see that a change of variable $y = \frac{1}{2}(x-\alpha)$ yields a function$g(x) = f(x)$defined for$-1 \le y1$ that has a turning point at zero and $\omega(g', \delta) \le 4\omega(f', \delta)$.

we assume $w.l.o.g. that\ g(0) = 0$ and flip g at zero at get $\hat{g}$. Note $that\ \omega(\hat{g}, \delta) \le 2\omega(g', \delta)$and that $\hat{g}\ has\ s-1$ changes of monotonicity. Using the lemma and the inductive hypothesis there exist a sequence$\{h_n\}_{n=2N(s-1)}^{\infty}$ of comonotone approximations to g . Inverting the change of variable the sequence $\{p_n(x)\}, p_n(x) = h_n(y)$ verifies the proposition for s.

3- **The Main Results**

Here let us introduce our main theorems. In the first theorem we see that the positivity constrain restrict very much the order of the modulus of smoothness. It mean one only.

Theorem3.1.

There exists an absolute constant $C = C(r)$ such that for every $f \in [-1,1]$ which alternates in sign r times in $[-1,1], 0 < r < \infty$, and each $n > 1$,

$$\bar{E}_n(f) \le C\omega\left(f, \frac{1}{n}\right), \tag{1}$$

when f is differentiable we expect to get higher order Jackson estimates. $f$changes sign at$-1 < y < 1\ if\ f(y) = 0$ and if for some$\varepsilon > 0, f(x)f(y) \le 0$ for all $y - \varepsilon < x_1 \le y \le x_2 < y + \varepsilon$. Such a y is called an alternation point of f. If f is continuously differentiable, then either $f'(y) = 0\ or\ f'(y) \ne 0$. In the latter case we will say that y is a point of proper alternation. Our estimates will depend on the location of the points of proper alternation. How large n should be will depend on f.

By setting $f(x) = \int^x g$, whereupon f is piecewise monotone and p'_n is copositive with f.

In our second theorem we strength our first theorem and show that theorem 3.1 cannot be proved in terms of $\omega_2(f, n^{-1})_{p,w}$.

Theorem3.2.

For any $n \ge 4, a > 0\ ad\ any\ 0 < p < 1$, there exist a set $f \in \triangle^0 \cap L_{p,w}$ such that any polynomial $p_n \in \triangle^0 \cap \prod_n$ satisfies

$$||f - p_n||_{p,w} > Aw_2(f, n^{-1})_{p,w} \tag{2}$$

Proof.

Let $y_1 = \{0\}$. For each n>4, A>0. Let us choose 0<b<1 from the condition

$$\frac{4 - b^2n^2}{cbn^2} = 8A$$

Where C is nonnegative constant.

Put

$$f(x) := \begin{cases} (x^2 - b^2)^2, & |x| \ge b \\ 0, & |x| < b \end{cases}$$

Let $p_n \in \prod_n \cap \triangle^0$ that is $p^1{}_n(x) \ge 0, x \in [0,1]$

Put

$$S_4(x) := (x^2 - b^2)^2,$$

and put

$$M_n := p_n - S_n.$$

Since $p''_n(0) \ge 0, then$

$$M''_n(0) + S''_4(0) = P''_n(0) \ge 0, \tag{3}$$

and

$$S_4(x) := (x^2 - b^2)^2$$
$$S_4'(x) = 4x(x^2 - b^2).$$
$$S_4''(X) = 8x + 4(x^2 - b^2)$$
$$S_4''(0) = 4(0)(2.0) + (0^2 - b^2).4$$
$$S_4''(0) = -4b^2.$$

and

$$M''_n(0) \ge -S''_4(0) = 4b^2.$$

Let us make use of S.N. Bernstein inequality to obtain

$$4b^2 \le M_n''(0) \le n^2 \sup_{x \in I} |M_n(x)|$$

Whence

$$4b^2 \leq n^2 \sup_{x\in I} |M_n(x)|$$

$$4b^2 \leq n^2 \sup_{x\in I} |p_n(x) - S_4(X)|$$

$$\frac{4b^2}{n^2} \leq \sup_{x \in I} | p_n(x) - S_4(x)|$$

$$\leq \sup_{x \in I} |p_n(x) - f(x) + f(x) - S_4(x)|$$

$$\leq \sup_{x\in I} |p_n(x) - f(x)| + \sup_{x\in I} |f(x) - S_4(x)|$$

$$\frac{4b^2}{n^2} = \sup_{x\in I} |p_n(x) - f(x)| + b^4$$

So $$\frac{cn}{n} |||p_n - f||_{p,w} \geq \sup_{x\in I} |p_n(x) - f(x)|$$

$$\geq \frac{4b^2}{n^2} - b^4.$$

On the other hand

$$w_3(f, n^{-1}) \leq 2^{\frac{1}{p}} C(p)\, n^{-1} w_2(f', n^{-1}) \tag{4}$$

$$=2^{\frac{1}{p}} C(p)\, n^{-1} w_2(f' - S_4', n^{-1})$$

$$\leq 8\, C(p) n^{-1} \|f' - S_4'\|$$

$$\leq C(p). \frac{2^6 . b^3}{3\sqrt{3}} . n^{-1}$$

$$\leq C(p)\, 2^4\, b^4 n^{-1}$$

Hence

$$\frac{||f - p_n||_{p,w}}{w_2(f, n^{-1})_{p,w}} \geq \frac{\frac{4}{n^2} - b^2}{C(p)\, 2^3 b^3}$$

$$= \text{A}$$

Conclusions

The positivity of approximation like the target function restrict the degree of positive approximation in terms of $\omega_{\varphi}^{\,\prime}(f,\delta)_{p,\omega}$ and coin not replaced by $\omega_{\varphi}^{\,2}(f,\delta)_{p,\omega}$.

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