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Existence and uniqueness of solution for a semilinear wave equation with local dissipation

César Castañeda-Campos[1*], Luis Alberto Lara-Romero[2], Elmer René Chávez-Araujo[3], Víctor Guillermo Sánchez-Araujo[3], Javier Camilo Poma-Palacios[4], Mabel Yésica Escobar-Soldevilla[3], Emérita Heras-Girón[1]

[1] Graduate School. Universidad César Vallejo. Trujillo, Peru.

[2] Faculty of Physical and Mathematical Sciences. Universidad Nacional de Trujillo. Peru.

[3] Faculty of Engineering Sciences. Universidad Nacional de Huancavelica. Peru.

[4] Faculty of Electronics – Systems. Universidad Nacional de Huancavelica. Peru.

ABSTRACT

In the present article, by means of a process of suitable changes in the raised system we obtain a problem of non-homogeneous Cauchy Abstract and applying the theory of linear semigroups we obtain the existence and uniqueness of solution to the problem.

Keywords: Existence and uniqueness of solution, semi lineal problem, theory of semigroups.

INTRODUCTION

The term dissipative in an equation, serve to ensure that the energy associated with the equation has an asymptotic or polynomial behavior and makes sense the problem otherwise it can have explosion in a certain time.

Thus, in 1990, E. Zuazua, studied the equation of semilinear wave with localized dissipation in, bounded domain, where you have the "local dissipative term" and the semilinear term, expressed in [31]$\Omega \subset \mathbb{R}^n a(x)u_t f(u)$

$$\left| \begin{array}{lll} u_{tt} - \Delta u + \alpha u + f(u) + a(x)u_t = 0\,, & & in \quad \Omega \times (0,+\infty) \\ u(0) = u_0 \quad , & u_t(0) = u_1 & in \quad \Omega \\ u = 0 & & over \quad \Gamma \times (0,+\infty) \end{array} \right.$$

Where ω is a border vicinity of $\Omega \subset$, R - n. With these mentioned conditions it is proved that the problem is well placed in space, that is, for initial data $\Omega \subset \mathbb{R}^n H_0^1(\Omega) \times L^2(\Omega)$,

$\{u_0, u_1\} \in H_0^1(\Omega) \times L^2(\Omega)$, there is only one weak solution in the class,

$$u \in C\big([0,+\infty); H_0^1(\Omega) \cap H^2(\Omega)\big) \cap C^1\big([0,+\infty); L^2(\Omega)\big)$$

Many authors work with local damping, see [3, 6, 10, 11, 12, 13, 14] and [15]

The wave equations describe particle propagation. From a mathematical point of view the wave equation is the exact opposite of that of heat because it is a reversible system in time, conservative, devoid of regularizing effects and in which the speed of propagation is finite. They are applied to quantum mechanics (the Schrödinger equation, which represents the motion of microscopic particles, making a role analogous to Newton's second law in classical mechanics), physics (elasticity problems) and engineering (vibrations of structures, construction of bridges), among others, as described in the following figure.

Figure 1. Akashi Kaikyo Bridge, Japan

Figure 2. Structure of optical crystals via optical diffraction.

The local dissipative term $a(x)u_t$ in the proposed system causes the solution of the system to have an asymptotic behavior, that is, it decays exponentially when t tends to infinity and as a consequence the posed problem makes sense. The terms semilinear *f(u)* and *au* determine some more irregularities that affect the propagation of waves.

We study, one of its many problems proposed in the article by E. Zuazua [17], "Semilinear wave equation with local damping", given by

$$u_{tt} - \Delta u + \int_0^t g(t-\tau) div\{a(x)\nabla u(\tau)\} d\tau = f \;\;,\;\; in \;\; \Omega \times (0,+\infty)$$
$$u = 0 \qquad\qquad , \qquad\qquad over \;\; \Gamma \times (0,+\infty)$$
$$u(x,0) = u_0(x) \;\;, \qquad u_t(x,0) = u_1(x) \qquad , \;\; in \;\; \Omega$$

considering a real function of vector variable, that is, open and bounded, and with the hypotheses $a(x) a: \Omega \longrightarrow \mathbb{R}\,,\; \Omega \subset \mathbb{R}^n$

(H_1) $\;a \in L^\infty_+(\Omega)\,,\; a(x) \geq a_0 > 0\;, \quad c.s.\; en\;\; \omega \subset \Omega\;,\;\; \omega$ is an open neighborhood of $\;\Gamma = \partial\Omega$.

(H_2) $\;f(s)s \geq 0\;,\; \forall\, s \in\; \mathbb{R}$..

(H_3) $\;f \in C^1(\mathbb{R})$, satisfying the condition of growth, given in (1).

(H_4) $\;f\;$ is globally Lipschitziana, i.e. $f' \in L^\infty(\mathbb{R})$.

With these hypotheses the problem is well placed in space, that is, for any initial data and for, there is a single classical or regular solution in space. $(02) H^1_*(\Omega) \times L^2(\Omega) (u_0, u_1) \in H^1_*(\Omega) \times H^2(\Omega) \times L^2(\Omega)$

(03) $\quad u \in C\big([0,T]; H^1_*(\Omega)\big) \cap C^1([0,T]; H^1_*(\Omega) \times H^2(\Omega)\,) \cap C^2\big([0,T]; L^2(\Omega)\big)$

$$H_*(\Omega) = \left\{ u \in \; H^1(\Omega) \;/\; u + \frac{\partial u}{\partial v} = 0 \;,\; over \;\; \Gamma \;\right\}$$

That we will solve using the theory of linear semigroups.

The energy associated with the system (0.1), is formally given by

$$E(t) = \frac{1}{2}\left\{ \int_\Omega [|u_t(x,t)|^2 + |\nabla u(x,t)|^2] dx + 2\int_\Omega F\big(u(x,t)\big) dx + \int_\Gamma |u(x,t)|^2 d\Gamma \right\}$$

where

$$F(s) = \int_0^s |f(t)| dt \;\;,\;\; \forall\, s \in \mathbb{R}$$

It is shown that,

$E(t) \leq C E(0) e^{-\gamma t}\,,\;\; \forall\, t > 0\;,\;\; \gamma > 0$

For information on Sobolev spaces see [1, 2, 5], and on the theory of linear semigroups see [7, 8, 9].

1. Abstract Semilinear Problem

Consider the abstract initial value problem given by,

$$\begin{cases} \dfrac{du}{dt} = Au(t) + F(u(t))\,;\;\; t > 0 \\ u(0) = u_0\,, u_0 \in D(A) \end{cases} \qquad (1.3)$$

where is the generator of a contraction semigroup over a Banach space and a continuous function $\;AH \quad F: H \to H$

Definition 1

We will say that it is a classical or strong solution of the initial value problem (1.8) if

$u \in C^1([0,T];H) \cap C([0,T];D(A))$.

As $F: H \to H$ is continuous, integrating from 0 to t you have

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds$$

Definition 2

that an application is locally Lipschitzian, if for every positive constant there exists a constant that $F: H \to H \quad M \quad L_M$

$|F(u) - F(v)| \leq L_M |u - v|$

For everything, such that $u, v \in H \;\; |u| \leq M \quad y \quad |v| \leq M$

Motto 3

$T > 0 \;\; and \;\; u_0 \in H$. Si $u, v \in C([0,T];H)$ are two solutions of (1.9), then $u = v$

Demo. As and are two solutions of (1.4) thenuu

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds$$

$$v(t) = S(t)u_0 + \int_0^t S(t-s)F(v(s))ds$$

Later

$$|u(t) - v(t)| = \left| \int_0^t S(t-s)\left(F\big(u(s)\big) - F\big(v(s)\big)\right) ds \right|$$

$$\leq \int_0^t |S(t-s)| \left|F\big(u(s)\big) - F\big(v(s)\big)\right| ds$$

$$\leq L_M \int_0^t |u(s) - v(s)| ds$$

and by Gronwall's inequality we get $u = v$.

Theorem 5

u is a maximal solution of, then(1.3)

$T_{max} = +\infty$ ó $T_{max} < +\infty$ and

$\lim_{t \to t_{max}} |u(t)| = +\infty$

In the first case, we will say that it is a global solution and in the second case, we will say that the solution explodes in finite time.

Demo. See Peña Miranda C. [14].

2. Existence and uniqueness of the regular solution

En este capítulo estudiaremos la existencia y unicidad de la solución regular de la ecuación (02) con las hipótesis $\Omega \subset \mathbb{R}^n$ abierto limitado y bien regular $(H1) - (H4)$ y datos iniciales $(u_0, u_1) \in \Omega$

$$H^1(\Omega) \cap H^2(\Omega) \times H^1(\Omega)$$

2.1 Existence and uniqueness

Theorem 6

Let $(u_0, u_1) \in \Omega \subset \mathbb{R}^n$ that check the conditions Then there is a single solution $(0.2) - (0.5)$.

$u: [0, T_{max}[\to \mathbb{R}$ tal que

$u \in C\big([0, T_{max}[; H^2(\Omega)\big) \cap C^1\big([0, T_{max}[; H^1(\Omega)\big) \cap C^2\big([0, T_{max}[; L^2(\Omega)\big)$

Demonstration

Existence of the regular solution

Be the system

$$\begin{cases} u_{tt} - \Delta u + f(u) + a(x)u_t = 0 \ \ en \quad Q = \Omega \times (0; +\infty) \\ \dfrac{\partial u}{\partial v} + u = 0 \quad sobre \quad \Sigma = \Gamma \times (0; +\infty) \\ u(x, 0) = u_0(x, 0) \in H_0^1(\Omega) \\ u_t(x, 0) = u_1 \in L^2(\Omega) \end{cases}$$

Do $v = u_t$

$$\begin{cases} u_t = v \\ v_t = u_{tt} = \Delta u - f(u) - a(x)u_t \end{cases}$$

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} v \\ \Delta u \end{pmatrix} + \begin{pmatrix} 0 \\ -f(u) - a(x)v \end{pmatrix}$$

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ -f(u) - a(x)v \end{pmatrix}$$

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} + \begin{pmatrix} 0 & -I \\ -\Delta & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ -f(u) - a(x)v \end{pmatrix}$$

Then you have the initial value problem

$$\left| \begin{array}{l} U_t + AU = F \\ U(0) = U_O \end{array} \right.$$

where

$$A = \begin{pmatrix} 0 & -I \\ -\Delta & 0 \end{pmatrix}, U_0 = \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, F(U) = \begin{pmatrix} 0 \\ -f(u) - a(x)v \end{pmatrix}$$

Be $X = H^1(\Omega) \times L^2(\Omega)$

$\begin{array}{l} A: X \longrightarrow X \\ \ \ U \longmapsto AU \end{array}$, A is a linear operator

where $U = \begin{pmatrix} u \\ v \end{pmatrix} \in X \Rightarrow u \in H^1(\Omega), u_t = v \in L^2(\Omega)$

$AU = \begin{pmatrix} -v \\ -\Delta u \end{pmatrix} \in X \Rightarrow v \in H^1(\Omega), \Delta U \in L^2(\Omega), u \in H^2(\Omega)$

$D(A) = \left\{(u, v) \in H^2(\Omega) \cap H_0^1(\Omega) \times H^1(\Omega) : \ u + \dfrac{\partial u}{\partial v} = 0 \ \ in\ \partial\Omega\right\}$

Let's define the internal product in X

$$(U_1, U_2)_X = \int_\Omega (\nabla u_1 . \nabla u_2 + v_1 v_2) dx + \int_\Omega u_1 u_2 d\Gamma \ ; \ U_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, U_2 = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \in X$$

Statement 1: A it is monotonous

Indeed

It must be shown that $(AU, U) \geq 0, \ \forall\, U \in D(A)$

$$(AU, U)_X = \int_\Omega (-\nabla v . \nabla u - v\Delta u) dx + \int_\Gamma (-v) u d\Gamma$$

$$= \int_\Omega (-\nabla v . \nabla u + \nabla v . \nabla u) dx - \int_\Gamma v \frac{\partial u}{\partial v} d\Gamma - \int_\Gamma v u d\Gamma \ , (by\ Green)$$

$$= \int_{\Omega} (-\nabla v.\nabla u + \nabla v.\nabla u)dx - \int_{\Gamma} v\left(u + \frac{\partial u}{\partial v}\right)d\Gamma$$

$$= \int_{\Omega} 0dx - \int_{\Gamma} v.0d\Gamma$$

$$= 0 \geq 0 \quad ,U \in D(A) \ , \ u + \frac{\partial u}{\partial v} = 0$$

$\therefore A$ is monotonous

Statement 2: A it is maximal in $D(A) = D(I + A)$

Indeed

That is, it must be shown that

$\forall\, F \in X = H^1(\Omega) \times L^2(\Omega)\ , \exists\, U \in D(I + A)\ /\ (I + A)U = F$

$A = \begin{pmatrix} 0 & -I \\ -\Delta & 0 \end{pmatrix}, (I + A) = \begin{pmatrix} I & -I \\ -\Delta & I \end{pmatrix}$

$(I + A)U = \begin{pmatrix} I & -I \\ -\Delta & I \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u - v \\ -\Delta u + v \end{pmatrix}$

$F = \begin{pmatrix} f \\ g \end{pmatrix}$

then

$(I + A)U = F$

$$\left| \begin{array}{l} u - v = f \in H^1(\Omega) \\ -\Delta u + v = g \in L^2(\Omega) \end{array} \right.$$

Addend

$u - \Delta u = (f + g) \in L^2(\Omega)$

let's apply Lax–Milgram's theorem

It is also maximal if it satisfies that: A

$a) a(.,.)$ is bilinear

$b) a(.,.)$ is coercive

then

let's define the bilinear function

$a: H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}\ , u, v \in H_0^1(\Omega)$

$a(u, v) = (u - \Delta u, v)_{L^2(\Omega)} = (u, v) + (-\Delta u, v) - (u, v)_{L^2(\Gamma)}$

$$= \int_{\Omega} uvdx + \int_{\Omega} (-v\Delta u)dx - \int_{\Gamma} uvd\Gamma$$

$$= \int_{\Omega} uvdx + \int_{\Omega} \nabla u \nabla vdx - \int_{\partial\Omega} v\frac{\partial u}{\partial v}d\Gamma - \int_{\Gamma} uvd\Gamma \ \ (por\ Green)$$

$$= \int_{\Omega} uvdx + \int_{\Omega} \nabla u \nabla vdx - \int_{\Gamma} v\left(u + \frac{\partial u}{\partial v}\right)d\Gamma$$

$$= \int_{\Omega} uvdx + \int_{\Omega} \nabla u \nabla vdx - \int_{\Gamma} v.0d\Gamma$$

$$= \int_{\Omega} uvdx + \int_{\Omega} \nabla u \nabla vdx$$

$\therefore a(u, v)$ es bilineal

Affirmation 3: $a(.,.)$ it is coercive

We must show that $|a(u,u)| \geq c\|u\|_{H^1(\Omega)}^2, \forall u \in H_\partial^1(\Omega)$

Indeed

$$a(u,u) = \int_\Omega u.udx - \int_\Omega \nabla u.udx - \int_\Omega uud\Gamma$$

$$= \int_\Omega u^2dx + \int_\Omega (\nabla u)^2 - \int_\Omega u\frac{\partial u}{\partial v}d\Gamma - \int_\Gamma uud\Gamma$$

$$= \int_\Omega u^2dx + \int_\Omega (\nabla u)^2 - \int_\Omega u\left(u + \frac{\partial u}{\partial v}\right)d\Gamma$$

$$= \int_\Omega u^2dx + \int_\Omega |\nabla u|^2 - \int_\Omega u.0d\Gamma$$

$$= \int_\Omega |u|^2dx + \int_\Omega |\nabla u|^2dx$$

$$= \|u\|_{H^1(\Omega)}^2$$

$\therefore |a(u,u)| \geq \|u\|_{H^1(\Omega)}^2$

$\therefore A$ is maximal

Uniqueness of the regular solution

Let us consider and two solutions of the equation and let, then $uv(02)w = u - v$

$$\begin{cases} w_{tt} - \Delta w + f(u) - f(v) + a(x)w_t = 0 \; in \quad Q = \Omega \times (0,+\infty) \\ w(0) = w_t(0) \qquad\qquad over \quad \Sigma = \Gamma \times (0,+\infty) \end{cases}$$

$$\begin{cases} w_{tt} - \Delta w + a(x)w + f(u) - f(v) + a(x)w_t = 0 \\ w(0) = w_t(0) = 0 \end{cases}$$

Multiplying this equation by w_t and integrating on $\mathbb{R}^n$ it has

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |w_t|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla \mathrm{w}|^2\,dx + \int_\Gamma ww_t d\Gamma + \frac{d}{dt}\int_\Omega (f(u) - f(v))dx = -\frac{1}{2}\int_\Omega a(x)|w_t|^2\,dx$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |w_t|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla \mathrm{w}|^2\,dx + \frac{d}{dt}\int_\Omega (f(u) - f(v))dx + \frac{1}{2}\int_\Gamma |w|^2 d\Gamma = -\frac{1}{2}\int_\Omega a(x)|w_t|^2\,dx$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |w_t|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla \mathrm{w}|^2\,dx + \frac{1}{2}\int_\Gamma |w|^2 d\Gamma = \frac{d}{dt}\int_\Omega (f(u) - f(v))dx - \frac{1}{2}\int_\Omega a(x)|w_t|^2\,dx$$

$$\frac{d}{dt}\left(|w_t|_{L^2(\mathbb{R}^n)}^2 + |w|_{L^2(\mathbb{R}^n)}^2\right) \leq \int_\Omega |f(u) - f(v)||w_t|dx + |a|_\infty |w_t|_{L^2(\mathbb{R}^n)}^2$$

2.2. Prolongation of the regular solution

Here we will obtain the overall solution of the problem, applying the theorem. $(02)(1.4.22)$

Multiplying the equation by and integrating over, we obtain(02) u_t Ω

$(u_{tt} - \Delta u + f(u) + a(x)u_t; u_t)$

$= (u_{tt}, u_t) + (-\Delta u, u_t) + (f(u), u_t) + (a(x)u_t, u_t) = 0 \qquad (*)$

Developing $(-\Delta u, u_t)$

By Green

$$(-\Delta u, u_t) = \int_\Omega (-\Delta u u_t)dx = \int_\Omega \nabla u \nabla u_t dx - \int_\Gamma u_t \frac{du}{dv} d\Gamma$$

Developing $(f(u), u_t)$

$$(f(u), u_t) = \int_\Omega f(u)\, u_t(x,t) dx = \int_\Omega F'(u(x,t))\, u_t(x,t) dx$$

$$= \frac{d}{dt}\int_\Omega F(u(x,t)) dx$$

Replacing in $(*)$

$$(u_{tt}, u_t) + \int_\Omega \nabla u \nabla u_t dx - \int_\Gamma u_t \frac{du}{dv} d\Gamma + \frac{d}{dt}\int_\Omega F(u(x,t)) dx + (a(x)u_t, u_t) = 0$$

$$(u_{tt}, u_t) + \int_\Omega \nabla u \nabla u_t dx - \int_\Gamma u_t \frac{du}{dv} d\Gamma + \frac{d}{dt}\int_\Omega F(u(x,t)) dx + (a(x)u_t, u_t) = 0$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_t|^2\, dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\, dx + \int_\Gamma u u_t d\Gamma + \frac{d}{dt}\int_\Omega F(u(x,t)) dx = -\frac{1}{2}\int_\Omega a(x)|u_t|^2\, dx$$

$$\frac{d}{dt}\left[\int_\Omega \frac{1}{2}|u_t|^2\, dx + \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \int_\Omega F(u(x,t)) dx + \frac{1}{2}\int_\Gamma |u|^{\,2} d\Gamma\right] = -\int_\Omega a(x)|u_t|^2\, dx$$

We define energy $E(t)$ by

$$E(t) = \frac{1}{2}\int_\Omega |u_t|^2\, dx + \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \int_\Omega F(u(x,t)) dx + \frac{1}{2}\int_\Gamma |u|^{\,2} d\Gamma$$

Then integrating from t_1 a t_2

$$\int_{t_1}^{t_2} \frac{d}{dt}(E(t)) = -\int_{t_1}^{t_2}\int_\Omega a(x)|u_t|^2\, dx\, dt$$

$$E(t_2) - E(t_1) = -\int_{t_1}^{t_2}\int_\Omega a(x)|u_t|^2\, dx\, dt$$

whether $t_2 = T$ and $t_1 = 0$

$$E(T) - E(0) = -\int_{t_1}^{t_2}\int_\Omega a(x)|u_t|^2\, dx\, dt < 0$$

Then

$E(T) \leq E(0)$

This is

$E(t)$ has a decrease when $t \to +\infty$

We can then say that

$E(0) \geq E(t), \forall t \geq 0$

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