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# Filter in Topological simple ring

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**1.Abstract:**

A topological simple ring S has the algebraic structure of a ring and topological structure of a topological space. A filter is a power tool both in topology and set theory. Special type of filters called ultrafilters have many useful technical properties. Filters have generalizations called p-filters(filter bases) and filter subbases which appears naturally and repeatedly throughout topology.

**Keywords:** Filters, p- filters, topological simple ring.

## 2.Introduction:

This paper attributes the concept of filter on topological simple ring. Also here we elucidate examples and basic results related to a-sequentially converges via filter and p-filter.

The concept of topological ring was introduced by D. Van Dantzig and developed by S. Warner[5] .The concept of topological simple ring[3] was defined and their properties are studied. Connor and Grosse –Erdmann[1] have investigated the impact of changing the definition of the convergence of sequences the structure of sequential continuity of real functions. In this paper, S will always denote a topological simple ring written additively or multiplicatively which satisfies the first axiom of countability. The letter o, p, q denote the sequences o = $(o_m)$, p = $(p_m)$, q = $(q_m)$ …. . of terms of S. s(Sc(S) denote the set of all S - valued sequence and the set of all S - valued convergent sequence of point in S respectively. By a technique of sequential convergence or a technique, we mean an additive or multiplicative function a defined on subring of $c_a$(S) of s(S) into S. A sequence o = $(o_m)$ is said to be a-convergent to r if o ∈ $c_a$(S) and a(o) = r. In particular, lim o = $\lim_{m\to\infty} o_m$ the simple ring c(S). A technique a is called regular if every convergent sequence $(o_m)$ is a – convergent with a(o) = lim o. First of all, we recall the definition of a-sequential closure of a subset of S. Let R ⊆ S and r ∈ S. Then r is in a- sequential closure of T if there is a sequence o = $(o_m)$ of points in R such that a(o) = r. We denote a- sequential closure of a set R by $\bar{R}^a$.

We say that a subset R is a-sequentially closed if it contains all of the point in its a-sequential closure (i.e) a subset R of S is a-sequential closed if $\bar{R}^a \subseteq R$ . The null set ∅ and the whole space S are a-sequentially closed. It is clear that $\bar{\emptyset}^a = \emptyset$ and $\bar{S}^a = S$ for a regular method a. If a is regular method, then R⊆ $\bar{R} \subseteq \bar{R}^a$ and hence R is a-sequentially closed if and only if $\bar{R}^a = R$. The concept of a filter was introduced by Henri cartan[2] in 1937. In 2002, Preuss[4] has applied filters throughout his book on convenient topology. In 2002, Beattie and Butzmann described non-topological convergence notion in functional analysis. The study of filters is a very natural way to describe convergence in general topological space. More recently filters play a fundamental role in the development of fuzzy spaces which have application in computer science and engineering. Filters are nearly new in topological simple ring to characterize such significant concept as a- sequentially converges.

## 3.Topological Simple Ring

**Definition: 3.1**

A topological simple ring S is a simple ring which is also a topological space if the following conditions are satisfied:

(i)for each s, t∈ S and each open neighbourhood L of s-t in S, there exist open neighboourhood J of s and K of t in S such that J-K ⊆ L.

(ii)for each s, t∈ S and each open neighbourhood L of st in S, there exist open neighboourhood J of s and K of t in S such that JK⊆L.

**Example :3.2**

Let S = $\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | a,b,c,d \in Z_2\right\}$ be a simple ring under addition and multiplication we define a topology on S by T = $\left\{\emptyset, \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}, S\right\}$.

Now $S \times S = \left\{\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) | \, a,b,c,d \in Z_2\right\}$ and

$$\left\{\begin{matrix} \emptyset, \left\{\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right)\right\}, \left\{\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) | \, a,b,c,d \in Z_2\right\}, \\ \left\{\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) | \, a,b,c,d \in Z_2\right\}, S \times S \end{matrix}\right\}$$

. Clearly (i) and (ii) conditions in definition 3.1 are continuous. Therefore (S, +,. T)

## 4. Filter neighbourhood of identity

In this section, first we introduce a filter in topological simple ring and the following theorem gives necessary and sufficient conditions for a filter to be the filter neighborhoods of identity on S.

**Definition:4.1**

A filter is a non-empty collection $\mathcal{F}$ of subset of a topological simple ring S such that(i)$\emptyset \notin \mathcal{F}$ (ii)If M $\in \mathcal{F}$ and N$\supseteq M$, then N $\in \mathcal{F}$ (iii)If M $\in \mathcal{F}$ and N $\in \mathcal{F}$, then M $\cap N \in \mathcal{F}$.

**Example 4.2**

The set of all neighborhoods of a point s $\in$S in a filter is called the neighborhoods filter $\mathbb{N}$ of s.

**Theorem 4.3**

If $\mathbb{N}$ is the filter neighborhoods of identity i of S, then

(i) for each J$\in \mathbb{N}$, there exists K$\in \mathbb{N}$ such that -K $\subset$ J.

(ii) for each J$\in \mathbb{N}$ and s$\in$ J, there exists K$\in \mathbb{N}$ such that s+K$\subset$ J and K+s$\subset$J.

(iii) for each J$\in \mathbb{N}$ and s$\in$ J, there exists K$\in \mathbb{N}$ such that sK$\subseteq$J and Ks$\subseteq$ J.

(iv) for each J$\in \mathbb{N}$, there exists K$\in \mathbb{N}$ such that K+K $\subseteq$ J and KK $\subseteq$ J.

(v) for each J$\in \mathbb{N}$, there exists K$\in \mathbb{N}$ such that K-K $\subseteq$J.

**5.a-sequentially converges via filter**

In this section, we discuss their a-sequentially converges properties in filter.

**Definition 5.1:**

A subset $\mathbb{N}^a$ of S is called a-sequentially neighbourhood at a point s$\in$ S if there exists a-sequentially open set J with s $\in$ J$\subseteq$ $\mathbb{N}^a$ .

**Definition 5.2:**

The set of all a-sequentially neighborhoods of a point s $\in$ S in a filter is called the a-sequentially neighborhoods filter $\mathbb{N}^a$ of s.

**Definition 5.3:**

A filter $\mathcal{F}$ is said to a-sequentially converges to s$\in$S denoted by $\mathcal{F} \to s$ iff every a-sequentially neighbourhood of s belong to $\mathcal{F}$.

**Definition 5.4:**

If s$\in \bar{F}^a$ for every F$\in \mathcal{F}$, then s is a-sequentially closure or a-hull of $\mathcal{F}$.

**Theorem 5.5:**

Let R be a subset of S. Then for s$\in$ S, s $\in \bar{R}^a \Leftrightarrow$ there exist a filter on S and a- sequentially converges to s.

Proof:

Let s$\in \bar{R}^a$. Then any a- sequentially neighbourhood of s has a non-empty intersection with R and $\mathbb{N}^a \cap R$ form a filter base. Hence filter $\mathcal{F}$ is a- sequentially converges to s. Conversely we assume that $\mathcal{F}$ is a filter containing R and $\mathcal{F} \to s$. Then there exist a-sequentially neighbourhood $\mathbb{N}^a$ of s such that $\mathbb{N}^a \in \mathcal{F}$. Hence $\mathbb{N}^a \cap R \neq \emptyset$. So s$\in \bar{R}^a$.

**Theorem 5.6:**

Let S be a Hausdorff Topological simple ring $\Leftrightarrow$ each filter $\mathcal{F}$ a- sequentially converges to atmost one point.

Proof:

Let S be Hausdorff and $\mathcal{F} \to s,\ \mathcal{F} \to t$ where s $\neq t$. Then there exist ${\mathbb{N}_s}^a$ and ${\mathcal{M}_t}^a$ such that ${\mathbb{N}_s}^a \cap {\mathcal{M}_t}^a = \emptyset$, so $\mathcal{F} \to s \Rightarrow$ ${\mathbb{N}_s}^a \in \mathcal{F}$ and ${\mathcal{M}_t}^a \in \mathcal{F}$. Therefore ${\mathbb{N}_s}^a \cap {\mathcal{M}_t}^a = \emptyset \in$ $\mathcal{F}$ which is contradiction. Let s$\neq$t and ${\mathbb{N}_s}^a \cap {\mathcal{M}_t}^a \neq \emptyset$. Let $\mathcal{B}$ = $\{{\mathbb{N}_s}^a \cap {\mathcal{M}_t}^a / {\mathbb{N}_s}^a, {\mathcal{M}_t}^a \in S\}$. Now $({\mathbb{N}_s}^a \cap {\mathcal{M}_t}^a) \cap ({\wp_s}^a \cap {\mathbb{Q}_t}^a) = ({\mathbb{N}_s}^a \cap {\wp_s}^a) \cap ({\mathcal{M}_t}^a \cap {\mathbb{Q}_t}^a) \in \mathcal{B}$ . Since ${\mathbb{N}_s}^a \cap {\wp_s}^a$ is an open set containing s and ${\mathcal{M}_t}^a \cap {\mathbb{Q}_t}^a$ is an open set containing t. Thus $\mathcal{B}$ is a filter base. Since ${\mathbb{N}_s}^a \cap {\mathcal{M}_t}^a \subseteq {\mathbb{N}_s}^a$, $\mathcal{F} \to s$ and ${\mathbb{N}_s}^a \cap {\mathcal{M}_t}^a \subseteq {\mathcal{M}_t}^a$, $\mathcal{F} \to t$ which is contradiction. Thus S is a hausdorff.

**Theorem 5.7:**

Let S be a topological simple ring. A set J$\subseteq$ S is a-sequentially open $\Leftrightarrow$ if whenever

$\mathcal{F} \to s$ with s$\in$ J, we have J$\in \mathcal{F}$.

Proof: The proof follows from definition 5.3

**Theorem 5.8:**

Let S and R be topological simple ring with s$\in$ S and u:S$\to$ R. Then u is a- sequentially continuous at s $\Leftrightarrow$ whenever $\mathcal{F}$ is a filter such that $\mathcal{F} \to s, u(\mathcal{F}) \to u(s)$.

Proof:

Suppose u is a- sequentially continuous at s and $\mathcal{F} \to s$. Let $\mathbb{N}_a$ be a- sequentially neighbourhood of u(s). By a- sequentially continuity, there is a-sequentially neighbourhood $\mathcal{M}_a$of s such that $u(\mathcal{M}_a)$$\subseteq \mathbb{N}_a$. Since $\mathcal{M}_a \in \mathcal{F},\ u(\mathcal{M}_a) \in u(\mathcal{F})$. Hence $\mathbb{N}_a \in u(\mathcal{F})$ and $u(\mathcal{F}) \to u(s)$. Let $\mathcal{F} \to s \Rightarrow u(\mathcal{F}) \to u(s)$. Then $(\mathcal{M}_a) \to u(s)$, for each a- sequentially neighbourhood $\mathbb{N}_a$ of u(s), $\mathbb{N}_a \in u(\mathcal{M}_a(s))$. Then there exist $\mathcal{M}_a \in u(\mathcal{M}_a)$ such that $\mathcal{M}_a \subseteq \mathbb{N}_a$. Hence u is a- sequentially continuous at s.

**6. a-sequentially converges via p-filter**

In this section, we define a-sequentially converges in p-filter and elucidate their properties.

**Definition 6.1:**

Let $\mathcal{F}$ be a filter on topological simple ring S. A subset $\mathcal{B}$ is called a filter base(prefilter) for $\mathcal{F}$ on S $\Leftrightarrow$ $\mathcal{B}$ is set of subsets of S satisfying the condition a) $\mathcal{B} \neq \emptyset,\ \emptyset \neq \mathcal{B}$ b)$B_1, B_2 \in \mathcal{B} \Rightarrow$ there exist $B_3 \in \mathcal{B}$ such that $B_3 \subseteq B_1 \cap B_2$.It is denoted by $\mathcal{F}_P$.

**Example 6.2:**

Let S be a topological simple ring and R⊆S. Then the set of all open neighborhoods of R is a prefilter, whose associated filter is the neighborhood filter $\mathbb{N}_{\mathbb{R}}$ of R .

**Definition 6.3:**

A family $\{\mathcal{F}_{P_\alpha}\}_{\alpha\in\aleph}$ of p-filter on a set S is compatible if there exists a p- filter $\mathcal{F}_P \supseteq \bigcup_{\alpha\in\aleph}\mathcal{F}_{P_\alpha}$.

**Definition 6.4:**

Let $\mathcal{F}_P$ be a p- filter in a topological simple ring and let s be a point of S. We say that a-sequentially converges to s and write $\mathcal{F}_P \to s$ if $\mathcal{F}_P$ refines the a- sequentially neighbourhood filter ${\mathbb{N}_s}^a$ of s .(i.e) Every neighbourhood ${\mathbb{N}_s}^a$ of s contains an element L of $\mathcal{F}_P$ **.**

**Definition 6.5:**

Let $\mathcal{F}_P$ be a p- filter in a topological simple ring and let s be a point of S. We say that s is a-sequentially closure of $\mathcal{F}_P$ if $\mathcal{F}_P$ is compatible with a-sequentially neighbourhood filter ${\mathbb{N}_s}^a$ (i.e) If every element of $\mathcal{F}_P$ meets every a-sequentially neighbourhood of s.

**Proposition 6.6:**

Let a be sequential technique on S and $\mathcal{F}_P$ be a p- filter on S with associated filter $\mathcal{F}$ and let $\mathcal{F}_P \le \mathcal{E}_P$ be a refines p-filter.

i) If $\mathcal{F}_P$ a-sequentially converges to s, then s is a- sequential closure of $\mathcal{F}_P$ .

(ii) $\mathcal{F}_P$ a-sequentially converges to s$\Leftrightarrow$ $\mathcal{F}$ a-sequentially converges to s.

(iii) s is a- sequentially closure of $\mathcal{F}_P$ $\Leftrightarrow$ s is a-sequentially closure of $\mathcal{F}$.

(iv)If $\mathcal{F}_P$ a-sequentially converges to s, then $\mathcal{E}_P$ a-sequentially converges to s.

(v) If s is a-sequentially closure of $\mathcal{E}_P$ , then s is a-sequentially closure of $\mathcal{F}_P$ .

(iv) S is Hausdorff $\Leftrightarrow$ every p-filter on S a-sequentially converges to atmost one point.

Proof: Obvious.

**Proposition 6.7:**

Le a be sequential technique on S and $\mathcal{F}_P$ be a p-filter on S. Then following condition are equivalent:

(i) s is a-sequentially closure of $\mathcal{F}_P$

(ii) There exist a refinement $\mathcal{E}_P$ of $\mathcal{F}_P$ such that $\mathcal{E}_P$ a-sequentially converges to s.

Proof:

(i)$\Leftrightarrow$ (ii) If s is a-sequentially closure of $\mathcal{F}_P$ , there exist a p-filter $\mathcal{E}_P$ refining both $\mathcal{F}_P$ and ${\mathbb{N}_s}^a$. Then $\mathcal{E}_P$ is finer p-filter a-sequentially converges to s.

(ii) Since $\mathcal{E}_P \to$ s, by proposition 6.6(i), s is a-sequentially closure of $\mathcal{E}_P$ and since $\mathcal{E}_P \ge \mathcal{F}_P$ and proposition 6.6(v), s is a- sequentially closure of $\mathcal{F}_P$ .

**Proposition 6.8:**

Let a be sequential technique on S and R⊆ S. Then

(i)s∈ $\bar{R}^a \Leftrightarrow$ s $ is a-sequentially closure of the p-filter $\mathcal{F}_{P_R}$ = {R}

(ii) s∈ $\bar{R}^a$ $\Leftrightarrow$there exists a filter $\mathcal{F}$ such that R ∈ $\mathcal{F}$ and $\mathcal{F} \to s$.

(iii) s∈ $\bar{R}^a$ $\Leftrightarrow$ the p-filter $\mathcal{F}_{P_R}$ = {R} is compatible with a-sequentially neighbourhood filter ${\mathbb{N}_s}^a$ of s.

Proof:

(i)By definition of a-sequentially closure, every element of $\mathcal{F}_{P_R}$meets every

a- sequentially neighbourhood ${\mathbb{N}_s}^a$ of s $\Leftrightarrow$ s ∈ $\bar{R}^a$. Proof of (ii) and (iii) is similar to (i)

**Lemma 6.9:**

Let a be sequentially technique on S. If ultra p-filter $F_{P_u}$ has as a- sequentially closure, then $F_{P_u} \to$ s.

Proof:

If $F_{P_u}$ has a-sequentially closure, then there exist $E_{P_u}$ which refines both $F_{P_u}$ and ${\mathbb{N}_s}^a$. Since $F_{P_u}$ is ultra, it is equivalent to all of its refinement, so $F_{P_u}$ itself refine ${\mathbb{N}_s}^a$. Hence $F_{P_u} \to$ s.

**Theorem 6.10:**

Let a be sequential technique on S. The followings are equivalent:

(i) For every family $\{F_{P_\alpha}\}_{\alpha\in\aleph}$ of a- sequentially closed subsets satisfying the finite intersection property $\bigcap_{\alpha\in\aleph} F_{P_\alpha} \neq \emptyset$

(ii) Every p-filter on S has a- sequentially closure.

(iii) Every ultra p- filter on S is a- converges.

Proof:

(i)⇒ (ii) Let $F_{P_\alpha} = \{L_\alpha\}$ be a p-filter on S the set $\{L_\alpha\}$ satisfy the finite intersection property, s ∈ $\bigcap_{\alpha\in\aleph}\overline{L_\alpha}$ and each $L_\alpha$ meets each neighbourhood ${\mathbb{N}_s}^a$ of s.

(ii) ⇒(iii) by lemma 6.9

(iii)⇒(i)Let $\{F_{P_\alpha}\}$ be a family of a- sequentially closed subset of S satisfying the finite condition. Then $\{F_{P_\alpha}\}$is a filtersubbase, so that there exist some ultra p-filter refining $F_{P_\alpha}$. By our assumption, there exist s ∈ S such that $F_{P_u} \to$ s and s is a-sequentially closure of $F_{P_u}$. So each $F_{P_\alpha}$ meets every neighbourhood of s. Thus s ∈ $\overline{F_{P_\alpha}} = F_{P_\alpha}$. Hence $F_{P_\alpha}$ is non-empty.

**Proposition 6.11:**

Let k: S → R be a function. The following are equivalents:

(i) k is a-sequentially continuous.

(ii) For every p- filter L on S a- sequentially converges at s, k(L) a- sequentially converges k(s).

(iii)For every p- filter L on S with a-sequentially closure, k(L) has a-sequentially closure.

Proof: By property of a- sequentially continuous, the above conditions are equivalent.

**Conclusion:**

In this paper, we evolved filter and p-filter in topological simple ring. The notion is further inculcate with properties and theorems. Also we introduced a perception of a-sequential converges via filter and p-filter and investigated interesting result and theorem explicated throughout the paper.

**References:**


[1] J.Connor, K.G. Grosse-Erdmann,Sequential definition of continuity for real function, Rocky Mountain J.Math, 33, No.1, 93-121, 2003.

[2] Henri Cartan, Theorie des filtres, C. R. Acad. Sci.,, Paris, 595-598, 1937.

[3] U.Jerseena, S.Syed Ali Fathima, and K.Alli,Some salient feature of topological simple ring, Malaya journal of Matematik vol.8, 4, (1912-1915), ISSN(2319 -3786) ,2020.

[4] G.Preuss, Foundations of topology, An approach to Convenient topology, Kluwer Academic Publishers, Dordrecht, 2002.

[5] S.Warner, Topological Rings, North Holland Mathematics Studies, Elsevier, Amsterdam, London , New York , Tokyo, Vol.178,1993.