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Group$\{1, -1, i, -i\}$Cordial Labeling in Some Duplicate Graphs

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Abstract: We present algorithms in this article and demonstrate the existence of Group $\{1, -1, i, -i\}$ Cordial Labeling in the extended duplicate graphs of complete bipartite, quadrilateral snake, alternate quadrilateral snake and duplicate graph of complete graph with even vertices, triangular snake, alternate triangular snake .

Key words: graph labeling, duplicate graph, Group$\{1, -1, i, -i\}$ Cordial graph.

1. INTRODUCTION

The graph labeling was proposed by Rosa [1] in 1967. A mapping of integers or values to the vertices(nodes) or edges(lines) or both subject to particular condition(s) is termed as graph labeling. If the collection of set of vertices (or edges) is the mapping's domain, then the labeling is called a vertex (or an edge) labeling. Various graph labeling have been used in the interim in over 2000 papers[3]. E. Sampath kumar initiated work of duplicate graphs and demonstrated number of findings[4]. K. Thirusangu, P.P Ulaganathan and P. Vijaya kumar established prime and total cordiality in ladder graph's duplicate graph [5]. Athisayanathan et al [2] made an attempt to link concept of graph labeling using elements of a group $\{1, -1, i, -i\}$ cordiality in some classes of graphs. In this paper, we apply the same labeling in certain classes of duplicate and extended duplicate of graphs.

2. PRELIMINARIES

Definition 2.1: A group is a non-empty set along with a binary operation satisfying closure, associative, identity and inverse axioms. Order of an element a is the least positive integer n having $a^n = e$ where e is group's identity.

Definition 2.2: Group $\{1, -1, i, -i\}$ cordial labeling is assignment of the elements $1, -1, i, -i$ to the vertices such that an edge uv is assigned label 1 if $gcd\{o[f(u)], o[f(v)]\} = 1$ otherwise label 0 satisfying condition that difference of total number of vertices having different labels differ by at most one and that of edges having label 0 and 1 differ by at most 1.

Definition 2.3: Consider a graph with vertex set V with p vertices and q edges. We construct the duplicate graph as follows: For the vertex set V, there corresponds a vertex set V' and duplicate graph has vertex set $V \cup V'$ such that V and V' are disjoint and each vertex v in V corresponds to a unique vertex v' in V' in such a way that an edge uv in G corresponds to two edges uv' and $u'v$ in its duplicate graph and vice-versa.

3.MAIN RESULTS

Algorithm 3.1: Applying labels to extended duplicate graph's vertices of complete bipartite graph $EDG(K_{m,n})$

The vertices of duplicate graph of $K_{m,n}$ are $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n, u'_1, u'_2, \ldots, u'_m, v'_1, v'_2, \ldots, v'_n,$

Label the vertices as follows:

Case (i)$m < \left\lfloor \frac{m+n}{2} \right\rfloor$

Label the vertices $u_1, u_2, \ldots, u_m, v'_1, v'_2, \ldots, v'_{\left\lfloor \frac{m+n}{2} \right\rfloor - m}$ with 1

Label the vertices $v'_{\left\lfloor \frac{m+n}{2} \right\rfloor - m+1}, v'_{\left\lfloor \frac{m+n}{2} \right\rfloor - m+2}, \ldots, v'_n,$ with -1

Label the vertices $v_1, v_2, \ldots, v_{\left\lfloor \frac{m+n}{2} \right\rfloor - m}, u'_1, u'_2, \ldots, u'_m$ with i

Label the vertices $v_{\left\lfloor \frac{m+n}{2} \right\rfloor - m+1}, v_{\left\lfloor \frac{m+n}{2} \right\rfloor - m+2}, \ldots, v_n$ with $-i$

Case (ii) $m > \left\lfloor \frac{m+n}{2} \right\rfloor$

Label the vertices $v'_1, v'_2, \ldots, v'_n, u_1, u_2, \ldots, u_{\left\lfloor \frac{m+n}{2} \right\rfloor - n}$ with 1

Label the vertices $u_{\left\lfloor \frac{m+n}{2} \right\rfloor - n+1}, u_{\left\lfloor \frac{m+n}{2} \right\rfloor - n+2}, \ldots, u_m$ with -1

Label the vertices $v_1, v_2, \ldots, v_n, u'_1, u'_2, \ldots, u'_{\left\lfloor \frac{m+n}{2} \right\rfloor}$ with i

Label the vertices $u'_{\left\lfloor \frac{m+n}{2} \right\rfloor - n+1}, u'_{\left\lfloor \frac{m+n}{2} \right\rfloor - n+2}, \ldots, u'_m$ with $-i$

Case (iii) $m = \left\lfloor \frac{m+n}{2} \right\rfloor$ **then** $n = \left\lceil \frac{m+n}{2} \right\rceil$

Label the vertices $u_1, u_2, \ldots, u_m$ with 1

Label the vertices $u'_1, u'_2, \ldots, u'_m$ with -1

Label the vertices $v_1, v_2, \ldots, v_n$ with i

Label the vertices $v'_1, v'_2, \ldots, v'_n$ with $-i$

Theorem 3.1: The complete bi partite graph's extended duplicate graph EDG($K_{m,n}$) admits group $\{1, -1, i, -i\}$ cordial labeling.

Proof: The duplicate of complete bipartite graph has $2mn$ edges in the form of $u_i v'_j$ $(i = 1\ to\ m, j = 1\ to\ n)$, $u'_i v_j$ $(i = 1\ to\ m, j = 1\ to\ n)$

Case(i) $\boldsymbol{m < \left\lfloor \frac{m+n}{2} \right\rfloor}$

connect the vertices u_2, u_2' to get the extended duplicate graph

It is clear that $\left|V_f(x)\right| = \left\lfloor \frac{m+n}{2} \right\rfloor \ forall \ \ x = 1, -1, i, -i$

Now, for counting the number of edges labeled 1, we use inclusion-exclusion principle, the difference of the number of edges having one of the end vertices labeled 1 and total number of edges labeled 1 at both ends. $\left|e_f(1)\right| = mn + m\left(\left\lfloor \frac{m+n}{2} \right\rfloor - m\right) - m\left(\left\lfloor \frac{m+n}{2} \right\rfloor - m\right) + 1 = mn + 1$

Since the extended duplicate graph of complete bipartite graph has $2mn + 1$ edges

$\left|e_f(0)\right| = 2mn + 1 - mn - 1 = mn$ this guarantees that the labeling is group $\{1, -1, i, -i\}$ cordial.

Case(ii) $\boldsymbol{m > \left\lfloor \frac{m+n}{2} \right\rfloor}$

connect the vertices v_2, v_2' to get the extended duplicate graph

It is clear that $\left|V_f(x)\right| = \left\lfloor \frac{m+n}{2} \right\rfloor forallx = 1, -1, i, -i$

Proceeding as above, we have,
$\left|e_f(1)\right| = mn + n\left(\left\lfloor \frac{m+n}{2} \right\rfloor - n\right) - n\left(\left\lfloor \frac{m+n}{2} \right\rfloor - n\right) + 1 = mn + 1$ and

$\left|e_f(0)\right| = 2mn + 1 - mn - 1 = mn$

So, the labeling is group $\{1, -1, i, -i\}$ cordial.

Case(iii) $\boldsymbol{m = \left\lfloor \frac{m+n}{2} \right\rfloor}$ **then** $\boldsymbol{n = \left\lceil \frac{m+n}{2} \right\rceil}$

connect the vertices u_2, u_2' to get the extended duplicate graph

It is clear that $\left|V_f(1)\right| = \left\lfloor \frac{m+n}{2} \right\rfloor = \left|V_f(-1)\right|$; $\left|V_f(i)\right| = \left\lceil \frac{m+n}{2} \right\rceil = \left|V_f(-i)\right|$

$$\left|V_f(x) - V_f(y)\right| \leq 1 \quad for\ all\ x, y$$

$\left|e_f(1)\right| = mn + 1$ and $\left|e_f(0)\right| = 2mn + 1 - mn - 1 = mn$

So, the labeling is to be group $\{1, -1, i, -i\}$ cordial.

Algorithm 3.2: Applying labels to duplicate graph's vertices of Complete graph $\boldsymbol{DG(K_n)}$, $\boldsymbol{for\ \ n\ even}$

The vertices of duplicate graph of K_n are $v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n$

Label the vertices as follows: **when $\boldsymbol{n}$ is even**

$$f(v_k) = 1, for\ \ 1 \leq k \leq \frac{n}{2} \qquad f(v_k) = -1, for\ \ \frac{n}{2} + 1 \leq k \leq n$$

$$f(v_k') = i, for\ \ 1 \leq k \leq \frac{n}{2} \qquad f(v_k') = -i, for\ \ \frac{n}{2} + 1 \leq k \leq n$$

Theorem 3.2: The complete graph's duplicate graph $\boldsymbol{DG(K_n)}$, for $\boldsymbol{n}$ even, admits group $\boldsymbol{\{1, -1, i, -i\}}$ cordial labeling.

Proof: The duplicate graph of complete graph K_n has $n(n-1)$ edges in the format of

$v_i v'_j\ (i \neq j), v'_i v_j (i \neq j)\ \ for\ \ i, j = 1\ to\ n$

It is clear that $\left|V_f(x)\right| = \frac{n}{2} forall\ x = 1, -1, i, -i$. $\left|e_f(1)\right| = \frac{n}{2}(n-1)$ Since the duplicate graph has $n(n-1)$ edges $\left|e_f(0)\right| = \frac{n}{2}(n-1)$. So, the duplicate graph of K_n admits group $\{1, -1, i, -i\}$ cordial labeling when n is even.

Note: When n is odd, to be cordial $\left|V_f(x) - V_f(y)\right| \leq 1 \ \ for\ any\ x, y$

To satisfy the condition $\left|V_f(1)\right| = \left\lfloor \frac{n}{2} \right\rfloor or \left\lceil \frac{n}{2} \right\rceil$ that follows $\left|e_f(1)\right| = \left\lfloor \frac{n}{2} \right\rfloor (n-1)\ \ or \left\lceil \frac{n}{2} \right\rceil (n-1)$ and so, $\left|e_f(0)\right| = \left\lceil \frac{n}{2} \right\rceil (n-1) or \left\lfloor \frac{n}{2} \right\rfloor (n-1)$ hence $\left|e_f(1) - e_f(0)\right| = (n-1)\left(\left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{2} \right\rfloor\right) \geq n - 1$ which contradicts the definition of cordial. So, the duplicate graph of complete graph K_n, for n odd, does not admit group $\{1, -1, i, -i\}$ cordial labeling.

Algorithm 3.3: Applying labels to extended duplicate graph's vertices of Q$\boldsymbol{uadrilateral\ snake\ EDG(QS(n))}$

The vertices of duplicate graph of quadrilateral snake are $v_1, v_2, \ldots, v_{3n+1}, v'_1, v'_2, \ldots, v'_{3n+1}$

Label the vertices as follows:

$$f(v_k) = \begin{cases} -1, k\ being\ odd \\ -i,\ k\ being\ even \end{cases} \qquad f(v_k') = \begin{cases} i, k\ being\ odd \\ 1,\ k\ being\ even \end{cases}$$

Theorem 3.3: The Quadrilateral snake graph's extended duplicate graph admits group $\boldsymbol{\{1, -1, i, -i\}}$ cordial labeling.

Proof: The extended duplicate graph of quadrilateral snake has $8n + 1$ edges in the form of $v_{3i-2} v'_{3i+1}, v_{3i-2} v'_{3i-1}, v_{3i-1} v'_{3i}, v_{3i} v'_{3i+1}; v'_{3i-2} v_{3i+1}, v'_{3i-2}$

It is very obvious that $\left|V_f(x)\right| = \left\lfloor \frac{3n+1}{2} \right\rfloor or \left\lceil \frac{3n+1}{2} \right\rceil$so $\left|V_f(x) - V_f(y)\right| \leq 1\ \ forallx, y$

In this graph there are $\left\lfloor \frac{3n+1}{2} \right\rfloor$ vertices in the form v_k^i ($k\ being\ even$) and of them $\left\lfloor \frac{n}{2} \right\rfloor$ of them have degree 4 and remaining have degree 2 so the number of edges labeled 1 are

$$\left|e_f(1)\right| = 4 * \left\lfloor \frac{n}{2} \right\rfloor + \left(\left\lfloor \frac{3n+1}{2} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor\right) * 2 = \left(\left\lfloor \frac{3n+1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor\right) * 2 = 4n + 1$$

Since there are totally $8n + 1$ edges

$$\left|e_f(0)\right| = 8n + 1 - 4n - 1 = 4n$$

and $\left|e_f(1) - e_f(0)\right| \le 1$. Hence, the labeling is group $\{1, -1, i, -i\}$ cordial.

Algorithm 3.4: Applying labels to duplicate graph's vertices of triangular snake $DG(TS_n)$.

The nodes of duplicate graph of Triangular snake graph TS_n are $v_1, v_2, \dots, v_{2n+1}, v'_1, v'_2, \dots, v'_{2n+1}$

Label the vertices as follows:

$$f(v_k) = \begin{cases} 1, & for\ k \equiv 1\ mod\ 4 \\ -1, & for\ k \equiv 0,2\ mod\ 4 \\ i, & for\ \ k \equiv 3\ mod\ 4 \end{cases} \qquad f(v'_k) = \begin{cases} i, & for\ k \equiv 1\ mod\ 4 \\ -i, & for\ k \equiv 0,2\ mod\ 4 \\ 1, & for\ \ k \equiv 3\ mod\ 4 \end{cases}$$

Theorem 3.4: The triangular snake graph's duplicate graph admits group $\{1, -1, i, -i\}$ cordial labeling.

Proof: The duplicate graph of triangular snake has $6n$ edges in the form of $v_{2k}v'_{2k+1}$, $v'_{2k}v_{2k+1}$($for\ k = 1\ to\ n$); $v_{2k-1}v'_{2k+1}$, $v'_{2k-1}v_{2k+1}$ ($for\ k = 1\ to\ n$); $v'_{2k}, v'_{2k-1}v_{2k}(k = 1\ to\ n)$

$\left|V_f(-1)\right| = n = \left|V_f(-i)\right|$ and $\left|V_f(1)\right| = \frac{n+1}{2} + \frac{n+1}{2} = n + 1 = \left|V_f(i)\right|$

The number of edges having label 1 is depending on the number of vertices labeled 1. The vertices $v_k, for\ k \equiv 1,3\ mod\ 4$ are each of degree 4 except first and last vertex. The first and last such vertices are of degree 2. There are n+1 vertices $v_k, k \equiv 1 mod\ 4, v'_k \equiv 3\ mod\ 4$. Using inclusion-exclusion principle

$\left|e_f(1)\right| = 2 * 2 + 4(n + 1 - 2) - n = 3n$ and the remaining edges are labeled 0.

$\left|e_f(0)\right| = 6n - 3n = 3n$. So, the labeling is group $\{1, -1, i, -i\}$ cordial.

Algorithm 3.5: Applying labels to duplicate graph's vertices of alternate triangular snake $DG\big(A(TS_n)\big)$, $n \ge 2$

Case (i) When n is even

Label the vertices as follows:

$$f(v_k) = \begin{cases} 1, & if\ \ k\ being\ odd \\ i, & if\ k\ being\ even \end{cases}$$

$$f(v'_k) = \begin{cases} -1, & if\ \ k\ being\ odd \\ -i, & if\ k\ being\ even \end{cases}$$

Case (ii) When n is odd

Label the vertices as follows:

$f(v_{3k}) = 1,\ k = 1,2,3, \dots, n-1$ $\qquad f(v_{3n-2}) = 1$
$f(v_{3k-1}) = 1,\ k = 1,2,3, \dots, \frac{n-1}{2}$

$f(v'_{3k}) = -1,\ k = 1,2,3, \dots, n-1$ $\qquad f(v'_{3n-2}) = -1$
$f(v'_{3k-1}) = -1,\ k = 1,2,3, \dots, \frac{n-1}{2}$

$f(v_{3k-2}) = i,\ k = 1,2,3, \dots, n-1$ $\qquad f(v_{3n}) = i$
$f(v_{3k-1}) = i,\ k = \frac{n+1}{2}, \frac{n+3}{2}, \dots, n$

$f(v'_{3k-2}) = -i,\ k = 1,2,3, \dots, n-1$ $\qquad f(v'_{3n}) = -i$
$f(v'_{3k-1}) = -i,\ k = \frac{n+1}{2}, \frac{n+3}{2}, \dots, n$

Theorem 3.6 : The alternate triangular snake's duplicate graph , $DG\big(A(TS_n)\big)$, $n \ge 2$, admits Group $\{1, -1, i, -i\}$ cordial labeling.

Proof:

The duplicate graph of Alternate triangular snake, $DG\big(A(TS_n)\big)$, $n \ge 1$, has $6n$ vertices and $8n - 2$ edges.

Case(i): When n is even

By algorithm 3.5, it is clear that $\left|V_f(x)\right| = \frac{3n}{2}, for\ x = 1, -1, i, -i$

Since $\frac{3n}{2}$ odd indexed vertices v_k are labeled 1,and no edges has both end vertices with label 1. Of the above said vertices $(\frac{n}{2} + 1)$ of them are of degree 2 and remaining $(\frac{5n-1}{2})$ vertices are of degree 3.

$\left|e_f(1)\right| = (\frac{n}{2} + 1)$ *2+$\left(\frac{5n-1}{2}\right) * 3 = 4n - 1$

As the graph has total of $8n - 2$ edges remaining edges are labeled 0 and so $\left|e_f(0)\right| = 4n - 1$

Case(ii): When n is odd

By algorithm 3.5, it is clear that $\left|V_f(x)\right| = \frac{3n-1}{2}, for\ x = 1, -1$; $\left|V_f(x)\right| = \frac{3n+1}{2}, for\ x = i, -i$

Of these, vertices $v_{3k}, (k = 1\ to\ n), v_{3n-2}$ each of degree 3 are labeled 1 and vertices $v_{3k-1}, (k = 1\ to\ \frac{n-1}{2})$ each of degree 2 are labeled 1.

$$\left|e_f(1)\right| = (n-1) * 3 + 3 + \left(\frac{n-1}{2}\right) * 2 = 4n - 1$$

As the graph has total of $8n - 2$ edges remaining edges are labeled 0 and so $\left|e_f(0)\right| = 4n - 1$

Algorithm 3.7: Applying labels to extended duplicate graph's vertices of Alternate quadrilateral snake $EDG\big(A(QS_n)\big)$, $n \ge 2$

Label the vertices as follows:

$$f(v_k) = \begin{cases} 1, & if\ \ k\ being\ odd \\ i, & if\ k\ being\ even \end{cases}$$

$$f(v'_k) = \begin{cases} -1, & if\ \ k\ being\ odd \\ -i, & if\ k\ being\ even \end{cases}$$

Theorem 3.8: The alternate quadrilateral snake's extended duplicate graph, $EDG\big(A(QS_n)\big)$, $n \ge 2$, admits Group $\{1, -1, i, -i\}$ cordial labeling.

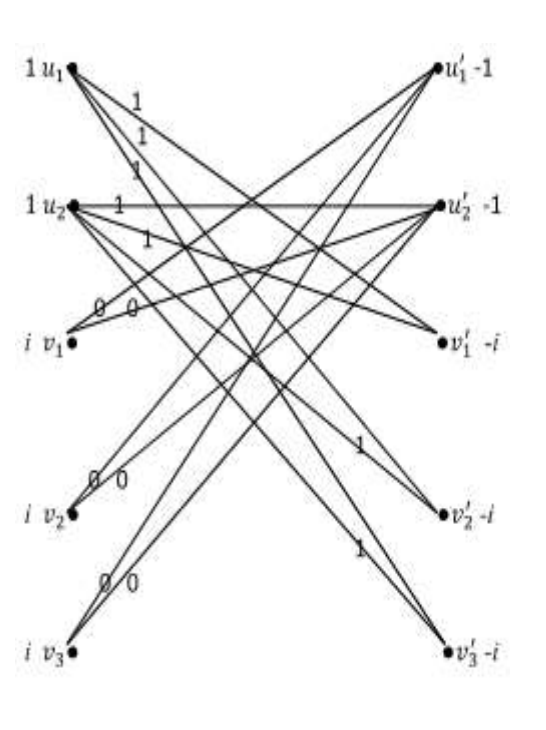
Proof: The extended duplicate graph of Alternate quadrilateral snake, $EDG\big(A(QS_n)\big)$, $n \ge 2$, has $8n$ vertices and $10n-1$ edges. By algorithm 3.7, $2n$ odd indexed vertices v_k are labeled 1 and no edges has both end vertices with label 1. Of the above said vertices $(n+1)$ of them are of degree 2 and remaining $(n-1)$ vertices are of degree 3.

$|e_f(1)| = (n+1)$ *2+$(n-1)*3 = 5n-1$

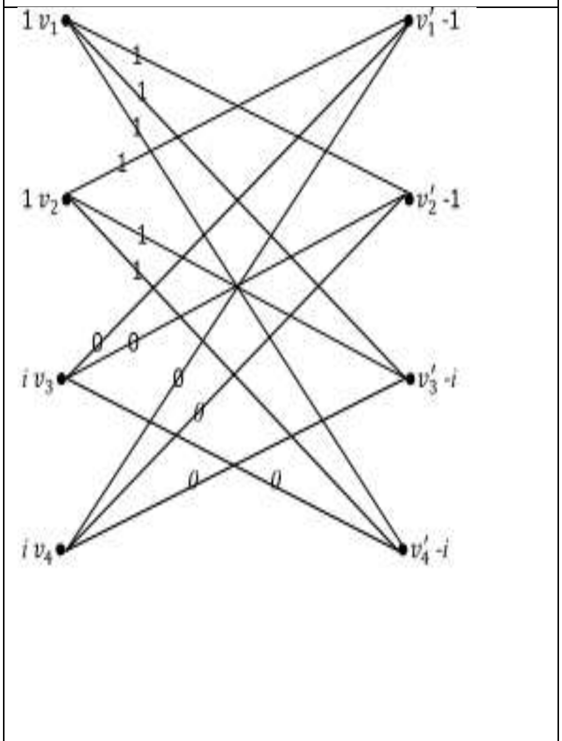
As the graph has total of $10n-1$ edges remaining edges are labeled 0 and so $|e_f(0)| = 5n$

ILLUSTRATION

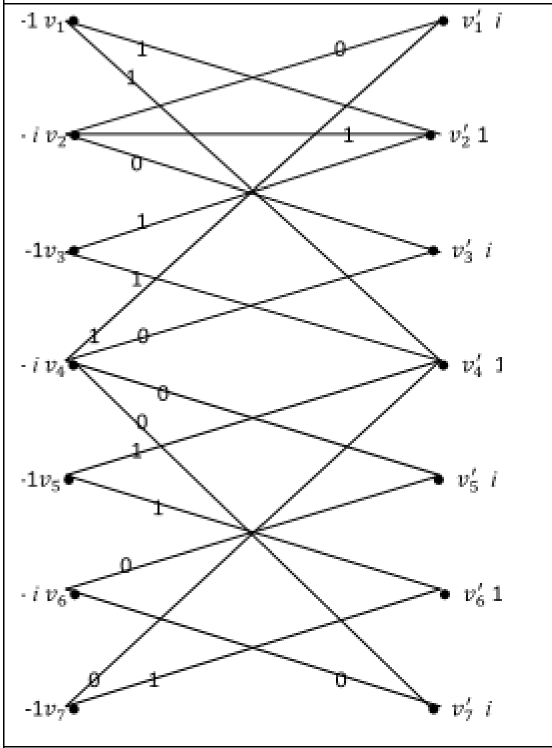
$\boldsymbol{EDG(K_{2,3})}$



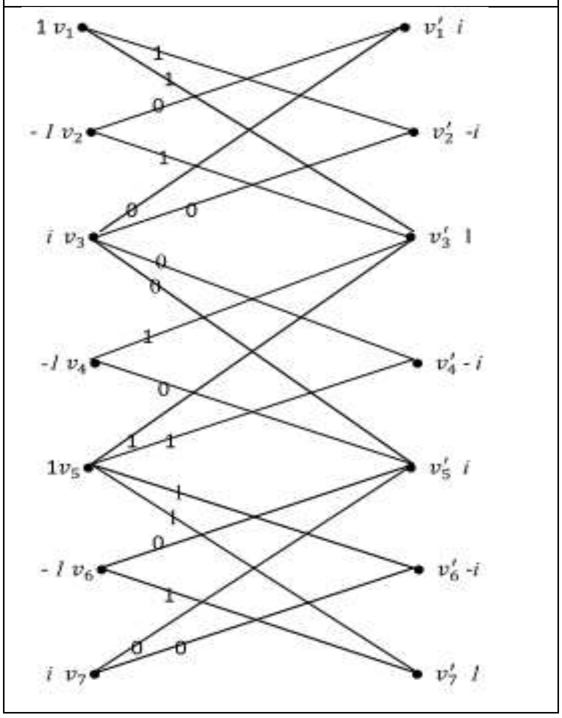
$\boldsymbol{DG(K_4)}$



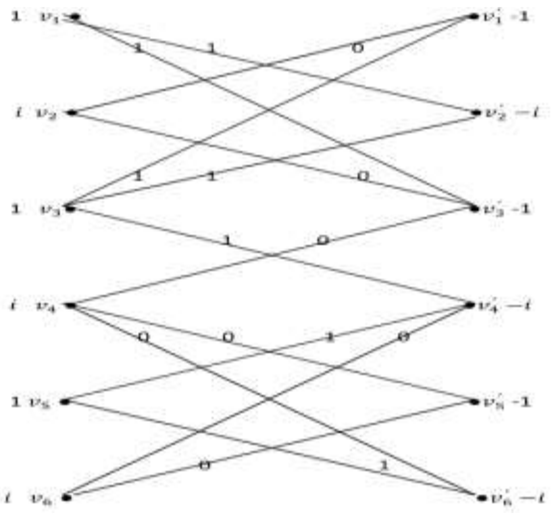
$\boldsymbol{EDG(QS_2)}$



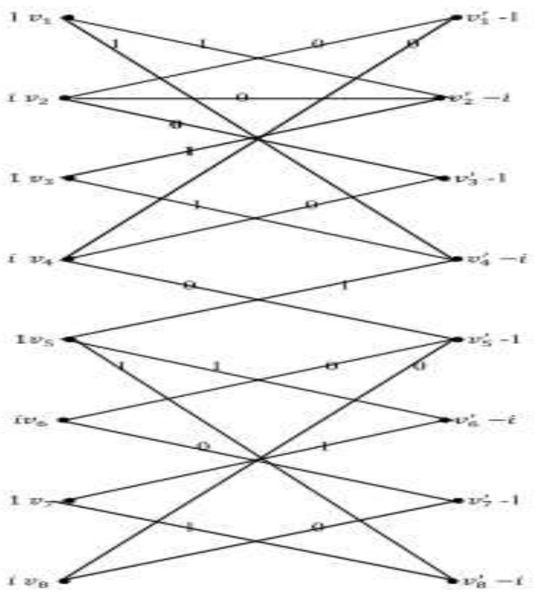
$\boldsymbol{DG(TS_3)}$



$DG(A(TS_2))$



$EDG(A(QS_2))$



4. CONCLUSION

We proved the existence of Group$\{1, -1, i, -i\}$ Cordial Labeling inthe extended duplicate graphs of complete bipartite graph, Quadrilateral snake, alternate quadrilateral snake and duplicate graph of complete graph with even number of vertices, triangular snake and alternate triangular snake.

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