ISSN: 0974-5823 Vol. 6 No. 3 December, 2021

International Journal of Mechanical Engineering

On the Evaluation of Modified Form of Gaussian Quadrature in Approximation of Numerical Integration

C. V. Rao

Department of General Studies, Jubail Industrial College, P. O. Box - 10099, Jubail Industrial City-31961, Saudi Arabia,

Abstract **- Gaussian quadrature is used to find the approximation of definite integral from *a* to *b* with integral *I* as a linear sum of $w_0 f(t_0), w_1 f(t_1), \dots w_n f(t_n)$ here Gaussian finds 2*n* + 2 unknowns with the degree of precision 2*n* + 1. Gaussian have chosen $f(t) = t^k$, $k \le 2n+1$ with the interval $[-1, 1]$, it gives $2n+2$ number of nonlinear equations in $2n+2$ number of variables. In this work, I replaced monomials t^k by trigonometric function $sin^k t$ with the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Here I discussed the two cases for *n* = 1 and *n* = 2 same as Gaussian. These results are significant to approximate the indeterminate integrals especially with trigonometric functions. I have also verified our results with some indeterminate integrals.**

Index Terms - Gaussian quadrature; definite integral; system of nonlinear equations.

INTRODUCTION

In order to find the approximate value of the definite integral $I = \int_a^b f(t)dt$ Gaussian [1],[2],[3] and [4] choose

$I_n(f) \approx \int_a^b f(t)dt$ ____________ (1)

Where $I_n(f) = w_0 f(t_0) + w_1 f(t_1) + \dots + w_n f(t_n)$ here t_i non-equally spaced points.

and w_i are weighted constants. Gaussian choose $t_0, t_1, \dots t_n$ and $w_0, w_1 \dots w_n$ simultaneously so that $I_n(f)$ has degree of precision as high as possible.

For $n = 1$ we want $\int_{-1}^{1} f(t)dt$

Consider $I_n(f) = w_0 f(t_0) + w_1 f(t_1)$

We have to choose t_0, t_1 w_0 and w_1 such that $I_n(f) = \int_a^b f(t)dt$, $f(x) = x^k$, $k \le 3$

So we have
$$\begin{cases} w_0 + w_1 = dt = 2 \\ w_0 t_0 + w_1 t_1 = \int_{-1}^{1} t\, dt = 0 \\ w_0 {t_0}^2 + w_1 {t_1}^2 = \int_{-1}^{1} t^2 dt = \frac{2}{3} \\ w_0 {t_0}^3 + w_1 {t_1}^3 = \int_{-1}^{1} t^3 dt = 0 \end{cases} \quad (1.1)$$

Solve this system of nonlinear equations to get t_0, t_1 w_0 and w_1

$$w_0 = 1, w_1 = 1, \; t_0 = -\frac{1}{\sqrt{3}} \text{ and } t_1 = \frac{1}{\sqrt{3}}$$

Thus $I_2(f) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

$$\int_{-1}^{1} (e^{2x} + 1)dx = I_2(e^{2x} + 1) = e^{\frac{2}{\sqrt{3}}} + e^{\frac{-2}{\sqrt{3}}} + 2 \approx 5.4882249$$

For $n = 2$ we want $\int_{-1}^{1} f(t)dt$

Consider $I_n(f) = w_0 f(t_0) + f(t_1) + w_2 f(t_2)$

We have to choose t_0, t_1, t_2, w_0, w_1 and w_2 such that $I_n(f) = \int_a^b f(t)dt$, $f(x) = x^k$, $k \le 5$

So we have
$$\begin{cases} w_0 + w_1 + w_2 = \int_{-1}^{1} 1\, dt = 2 \\ w_0 t_0 + w_1 t_1 + w_2 t_2 = \int_{-1}^{1} t\, dt = 0 \\ w_0 {t_0}^2 + w_1 {t_1}^2 + w_2 {t_2}^2 = \int_{-1}^{1} t^2\, dt = \frac{2}{3} \\ w_0 {t_0}^3 + w_1 {t_1}^3 + w_2 {t_2}^3 = \int_{-1}^{1} t^3\, dt = 0 \\ w_0 {t_0}^4 + w_1 {t_1}^4 + w_2 {t_2}^4 = \int_{-1}^{1} t^4\, dt = \frac{2}{5} \\ w_0 {t_0}^5 + w_1 {t_1}^5 + w_2 {t_2}^5 = \int_{-1}^{1} t^5\, dt = 0 \end{cases} \quad (1.1)$$

Solve this system of nonlinear equations to get t_0, t_1, t_2 w_0, w_1 and w_2

$$w_0 = \frac{5}{9}, \quad w_1 = \frac{8}{9}, \quad w_2 = \frac{5}{9},$$

$$t_0 = -\sqrt{\frac{3}{5}}, \; t_1 = 0 \text{ and } t_2 = \sqrt{\frac{3}{5}}$$

Thus $I_3(f) = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$

$$I = \int_{-1}^{1} (e^{2x} + 1)dx \approx$$

$$I_2(e^{2x}+1) = \frac{5}{9}e^{-\sqrt{\frac{3}{5}}} + \frac{8}{9}(2) + \frac{5}{9}e^{\sqrt{\frac{3}{5}}} + 2$$
$$I \approx 5.2392258$$

MODIFIED FORM OF GAUSSIAN QUADRATURE

In this case I replace the monomial x^k by $(sinx)^k$. This method is more useful find the approximate value of the definite integral involving trigonometric functions

$$I_n(f) \approx \int_a^b f(t)dt$$

to evaluate the definite integral $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t)dt$

$put \quad n = 1$

$$I_1(f) \approx \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t)dt$$

Consider $I_1(f) = w_0 f(t_0) + w_1 f(t_1)$

We have to find the constants t_0, t_1, w_0 and w_1 such that

$$I_n(f) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t)dt \quad , \quad f(t) = (sin\ t)^k \quad , k \le 3$$

for $k = 0,1,2\ and\ 3$ we get

$$\begin{cases} w_0 + w_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\ dt = \pi \\ w_0 sint_0 + w_1 sint_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin t\ dt = 0 \\ w_0 sin^2 t_0 + w_1 sin^2 t_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin t)^2\ dt = \frac{\pi}{2} \\ w_0 sin^3 t_0 + w_1 sin^3 t_1 = \int_{-1}^{1} (\sin t)^3 dt = 0 \end{cases}$$

ie.,

$$\begin{cases} w_0 + w_1 = \pi \\ w_0 sint_0 + w_1 sint_1 = 0 \\ w_0 sin^2 t_0 + w_1 sin^2 t_1 = \frac{\pi}{2} \\ w_0 sin^3 t_0 + w_1 sin^3 t_1 = 0 \end{cases}$$

Let $T_0 = sint_0$ and $T_1 = sint_1$ this gives

$$\begin{cases} w_0 + w_1 = \pi & \text{\_\_\_\_\_\_\_\_(2.1)} \\ w_0 T_0 + w_1 T_1 = 0 & \text{\_\_\_\_\_\_\_\_(2.2)} \\ w_0 {T_0}^2 + w_1 {T_1}^2 = \frac{\pi}{2} & \text{\_\_\_\_\_\_\_\_(2.3)} \\ w_0 {T_0}^3 + w_1 {T_1}^3 = 0 & \text{\_\_\_\_\_\_\_\_(2.4)} \end{cases}$$

Solve this system of nonlinear equations to get T_0, T_1, w_0 and w_1

(2.3)- T_1 ×(2.2)

Gives $w_0 T_0 (T_0 - T_1) = \frac{\pi}{2}$ ____________(2.5)

(2.4)- T_1 ×(2.3)

$w_0 {T_0}^2 (T_0 - T_1) = -\frac{\pi}{2}$ _____________(2.6)

ie., $T_0 = -1$

From (2.2) and (2.4)

$$w_1 T_1 = -w_0 T_0$$
$$w_1 {T_1}^3 = -w_0 {T_0}^3$$
$$({T_1}^2 - {T_0}^2) = 0 \Rightarrow ({T_1}^2 - {T_0}^2) = 0$$ (Since $w_1 T_1 \ne 0$)

$$\Rightarrow {T_0}^2 = {T_1}^2$$
$$\Rightarrow T_0 = \pm T_1$$
$$\Rightarrow T_0 = -T_1$$ (Since nodes are unequal)

Thus (2.1) and (2.1) become $w_0 + w_1 = \pi$

$$w_0 - w_1 = 0$$

Solving these two we get $w_0 = \frac{\pi}{2}$ and $w_1 = \frac{\pi}{2}$

From (2.3) $\frac{\pi}{2}{T_1}^2 + \frac{\pi}{2}{T_1}^2 = \frac{\pi}{2}$

$$\Rightarrow 2{T_1}^2 = 1$$
$$\Rightarrow {T_1}^2 = \frac{1}{2}$$
$$\Rightarrow T_1 = \frac{1}{\sqrt{2}} \quad \text{and} \quad T_0 = -\frac{1}{\sqrt{2}}$$
$$\Rightarrow t_0 = \frac{\pi}{4} \quad \text{and} \quad t_1 = -\frac{\pi}{4}$$
$$I_1(f) = w_0 f(t_0) + w_1 f(t_1)$$
$$\Rightarrow I_1(f) = \frac{\pi}{2} f\left(-\frac{\pi}{4}\right) + \frac{\pi}{2} f\left(\frac{\pi}{4}\right)$$

Ex 1: Evaluate: $I = \int_{-1}^{1} \frac{1}{\sqrt{1+x^3}} dx$

Sol:

Use the transformation $\pi x = (a+b) + t(b-a)$

$$\pi x = 2t$$
$$I = \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\sqrt{1+t^3}} dt$$

There fore $I_1(f) = \frac{2}{\pi}\left[\frac{\pi}{2} f\left(\frac{\pi}{4}\right) + \frac{\pi}{2} f\left(-\frac{\pi}{4}\right)\right]$

$$I_1(f) \approx 2.213509275$$

From Gaussian quadrature formula: $I = \int_{-1}^{1} \frac{1}{\sqrt{1+x^3}} dx$

$$I_1(f) \approx \boldsymbol{f}\left(\frac{-1}{\sqrt{3}}\right) + \boldsymbol{f}\left(\frac{1}{\sqrt{3}}\right)$$
$$I_1(f) \approx 2.028551636$$

From error function $I \approx 2.31179$. It means that modified method gives more accurate
result than Gaussian quadrature method

Put $n = 2$

$$I_2(f) \approx \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t)dt$$

Consider $I_2(f) = w_0 f(t_0) + w_1 f(t_1) + w_2 f(t_2)$

We have to find the constants t_0, t_1, t_1, w_0, w_1 and w_2 such that

$I_n(f) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t)dt \quad , \qquad f(t) = (\sin t)^k \quad , k \leq 5$

for $k = 0,1,2,3,4 \; and \; 5$ we get

$$\begin{cases} w_0 + w_1 + w_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \, dt = \pi \\ w_0 sint_0 + w_1 sint_1 + w_2 sint_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin t \; dt = 0 \\ w_0 sin^2 t_0 + w_1 sin^2 t_1 + w_2 sin^2 t_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin t)^2 \, dt = \frac{\pi}{2} \\ w_0 sin^3 t_0 + w_1 sin^3 t_1 + w_2 sin^3 t_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin t)^3 dt = 0 \\ w_0 sin^4 t_0 + w_1 sin^4 t_1 + w_2 sin^4 t_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin t)^4 dt = \frac{3\pi}{8} \\ w_0 sin^5 t_0 + w_1 sin^5 t_1 + w_2 sin^5 t_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin t)^5 dt = 0 \end{cases}$$

$$\begin{cases} w_0 + w_1 + w_2 = \pi \quad \_\_\_\_\_\_\_\_\_\_\_\_ (2.7) \\ w_0 T_0 + w_1 T_1 + w_2 T_2 = 0 \quad \_\_\_\_\_\_\_\_\_\_\_\_ (2.8) \\ w_0 {T_0}^2 + w_1 {T_1}^2 + w_2 {T_2}^2 = \frac{\pi}{2} \quad \_\_\_\_\_\_\_\_\_\_\_\_ (2.9) \\ w_0 {T_0}^3 + w_1 {T_1}^3 + w_2 {T_2}^3 = 0 \quad \_\_\_\_\_\_\_\_\_\_\_\_ (2.10) \\ w_0 {T_0}^4 + w_1 {T_1}^4 + w_2 {T_2}^4 = \frac{3\pi}{8} \quad \_\_\_\_\_\_\_\_\_\_\_\_ (2.11) \\ w_0 {T_0}^5 + w_1 {T_1}^5 + w_2 {T_2}^5 = 0 \quad \_\_\_\_\_\_\_\_\_\_\_\_ (2.12) \end{cases}$$

where $T_0 = sint_0, T_1 = sint_1,$ and $T_2 = sint_2$

From (2.8), (2.10) and (2.12)

$$(w_0 - w_2)\begin{bmatrix} T_0 \\ {T_0}^3 \\ {T_0}^5 \end{bmatrix} = -w_1 \begin{bmatrix} T_1 \\ {T_1}^3 \\ {T_1}^5 \end{bmatrix}$$

Here we have two options $w_0 - w_2 = 0$ and $w_1 = 0$ which is impossible

Since $w_i \neq 0 \; \forall \; i$

Or $w_0 - w_2 = 0$ and $T_1 = 0 \implies w_2 = w_0$ and $T_1 = 0$

Replace these values in (2.8), (2.10)

$$w_0 T_0 = -w_0 T_2$$
$$w_0 {T_0}^3 = -w_0 {T_2}^3$$

We get ${T_2}^2 = {T_0}^2$

$T_0 = \pm T_2 \implies T_0 = -T_2$ (Since variable points are non-equally spaced points)

From (2.9) and (2.11)

$$2 \, w_0 {T_0}^2 = \frac{\pi}{2} \quad \text{and} \quad 2 \, w_0 {T_0}^4 = \frac{3\pi}{8}$$
$${T_0}^2 = \frac{3}{4} \implies T_0 = -\frac{\sqrt{3}}{2} \quad \text{and} \quad T_2 = \frac{\sqrt{3}}{2}$$

From (2.7) and (2.9)

$$2w_0 + w_1 = \pi$$
$$2w_0 {T_0}^2 = \frac{\pi}{2} \implies w_0 = \frac{\pi}{3} \; , \; w_2 = w_0 = \frac{\pi}{3}$$
$$\text{and } w_1 = \pi - \frac{\pi}{3} = \frac{\pi}{3} \; ,$$

thus $w_0 = \frac{\pi}{3}$, $w_1 = \frac{\pi}{3}$ and $w_2 = \frac{\pi}{3}$

$T_0 = -\frac{\sqrt{3}}{2} \implies t_0 = -\frac{\pi}{3}$

$T_1 = 0 \implies t_1 = 0$ and

$T_2 = \frac{\sqrt{3}}{2} \implies t_2 = \frac{\pi}{3}$

$$I_2(f) = \frac{\pi}{3} f\left(-\frac{\pi}{3}\right) + \frac{\pi}{3} f(0) + \frac{\pi}{3} f\left(\frac{\pi}{3}\right)$$

Ex 2: Evaluate: $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-x^2} dx$

Sol: $I_2(f) = \frac{\pi}{3} f\left(-\frac{\pi}{3}\right) + \frac{\pi}{3} f(0) + \frac{\pi}{3} f\left(\frac{\pi}{3}\right)$

$$\implies I_2(f) \approx \frac{\pi}{3}\left(e^{-\frac{\pi^2}{9}} + 1 + e^{-\frac{\pi^2}{9}}\right)$$

$\implies I_2(f) \approx 1.746719622$. Which is more accurate than $I_1(f)$

From Gaussian quadrature formula $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-x^2} dx$

$$: \quad I = \frac{\pi}{2}\int_{-1}^{1} e^{-\frac{\pi^2 t^2}{4}} dt \approx \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

$$: \quad I = \frac{\pi}{2}\int_{-1}^{1} e^{-\frac{\pi^2 t^2}{4}} dt \approx 1.793391081$$

Transformation of variable according to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. For the integral $I = \int_a^b f(t)dt$ we need to transform the variable $t = \frac{1}{2\pi}\left(\pi(a+b) + \left(x - \frac{\pi}{2}\right)(b-a)\right)$.From error function

$$I \approx 1.7258$$

CONCLUSION

This modified Gaussian method is more accurate than Gaussian method. We can find the approximate value of indeterminate integral values more accurate than Trapezoidal, Simpson's rule and Bool's rule of integration. Modified form of Gaussian quadrature is also useful for indeterminate forms of definite integrals.

REFERENCES

[1] A. Townsend, *The race for high order Gauss–Legendre quadrature. SIAM News*, 48 (2015), pp. 1–3.

[2] Bogaert, *Iteration-free computation of Gauss–Legendre quadrature nodes and weights*, SIAM J. Sci. Comput., 36 (2014), C1008–C1026.

[3] C. F. Gauss, *Methodus nova integralium valores per approximationem inveniendi*, Comment. Soc. Reg. Scient. Gotting. Recent., (1814).

[4] G. H. Golub and J. H. Welsch, *Calculation of Gauss quadrature rules*, Math. Comp., 23 (1969), 221–230.

[5] I. Bogaert, B. Michiels, and J. Fostier, *O(1) computation of Legendre polynomials and Gauss–Legendre nodes and weights for parallel computing*, SIAM J. Sci. Comput., 34 (2012), pp. 83–101.

[6] Kahaner, David; Moler, Cleve; Nash, Stephen (1989), *Numerical Methods and Software*, Prentice–Hall, ISBN 978-0-13-627258-8.

[7] Laurie, Dirk (1997), "*Calculation of Gauss-Kronrod quadrature rules*.", Mathematics of Computation of the American Mathematical Society.

[8] Monegato, Giovanni (1978), "*Some remarks on the construction of extended Gaussian quadrature rules*", Mathematics of Computation, 32 (141): 247–252.

[9] N. Hale and A. Townsend, *Fast and accurate computation of Gauss–Legendre and Gauss–Jacobi quadrature nodes and weights*, SIAM J. Sci. Comput., 35 (2013), pp. A652–A674

[10] Piessens, Robert; de Doncker-Kapenga, Elise; Überhuber, Christoph W.; Kahaner, David K. (1983), QUADPACK, *A subroutine package for automatic integration*. Springer-Verlag, ISBN 978-3-540-12553-2 (Reference guide for QUADPACK).