Primitive Idempotents of Irreducible cyclic codes of length $32p^n$

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Abstract: Let $\mathbb{F}_q$ be a finite field of q elements such that $q \equiv 3(mod8)$ and p be an odd prime with $p^l \parallel q-1$ for integer $l > 0$ and $4 \nmid q-1$. In this paper, using matrix method, we intend to give all the $17p^n$ primitive idempotents in the ring $\mathbb{F}_q[x]/\left(x^{32p^n}-1\right)$ for two cases $n \le l \; and \; n > l$.

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1. Introduction

Let $\mathbb{F}_q$ be a finite field with q elements. Let $\mathcal{C}$ be a [m, k] linear code over $\mathbb{F}_q$, that is, it is a $k-$ dimensional sub space of $\mathbb{F}_q^m$. A code $\mathcal{C}$ is called cyclic if any cyclic shift given to a code word is again a code word. A code word $(c_0, c_1, c_2, \dots, c_{m-1})$ in $\mathcal{C}$ is identified with the polynomial $c_0 + c_1x + c_2x^2 + \dots + c_{m-1}x^{m-1}$ in $\mathbb{F}_q[x]/\langle x^m - 1\rangle$. In fact a code $\mathcal{C}$ of length m over $\mathbb{F}_q$ is a cyclic code if and only if the corresponding sub set is an ideal of $\mathbb{F}_q[x]/\langle x^m - 1\rangle$. Note that every ideal of $\mathbb{F}_q[x]/\langle x^m - 1\rangle$ is principal. Every cyclic code $\mathcal{C}$ of length m is generated by a unique monic divisor $g(x)$ of minimal degree in $\mathbb{F}_q$ and that $h(x) = (x^m - 1)/g(x)$ is referred to the parity check polynomial of $\mathcal{C}$. Irreducible cyclic code of length m over $\mathbb{F}_q$ can be viewed as ideals of the ring $\mathbb{F}_q[x]/\langle x^m - 1\rangle$ generated by the primitive idempotents.

A lot of papers and results in which the cyclic codes have been found: Arora and Pruthi [13,3], obtained the primitive idempotents in R_m for m =2,4, $l^n and \; 2l^n$, where l d is an odd prime and q (prime power).In [17] for $m = l_1^{m_1} l_2^{m_2}$; are distinct odd primes.$\left(\frac{\phi(l_1^n)}{2}, \frac{\phi(l_1^n)}{2}\right)$ $gcd\left(\phi({l_1}^{m_1}), \phi({l_2}^{m_2})\right) = 2. \; ord_{{l_1}^{m_1}}(q) = \frac{\phi({l_1}^{m_1})}{2}$ and $ord_{{l_2}^{m_2}}(q) = \frac{\phi({l_2}^{m_2})}{2}$. Bakshi and Raka [4] gave all the $3n+2$ primitive idempotents in the ring R_m for m =$l_1^n l_{2,}$, for distinct odd primes $l_1, l_{2,} and \; q$ and $gcd\left(\frac{\phi(l_1^n)}{2}, \frac{\phi(l_2)}{2} = 1\right)$.Singh and Pruthi gave explicit expressions for all the $4m_1m_2 + 2m_1 + 2m_2 + 1$ primitive idempotents in the ring R_m. In [9] , taking $m = 4l^n$ and $8l^n$, where l is an odd prime and $l \mid q-1$. Li and Yue et al. described all primitive idempotents and minimum Hamming distances of the codes generated by those primitive idempotents in R_m respectively. In [14], Pruthi and Pankaj found all the primitive idempotents in the ring $\mathbb{F}_l[x]/\langle x^{p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}} - 1\rangle$.In [15], Sehrawat and Pruthi gave explicit expressions for the idempotents in the group algebra of dihedral group of order $2n$, for every n.

This paper is organized as follows: In section 2 we recall some lemmas. In section 3, we give all primitive idempotents in $\mathbb{F}_q[x]/\langle x^{32p^n} - 1\rangle$ of irreducible cyclic codes of length $32p^n$.

2. Preliminaries

Following are the lemmas which give the criterion on irreducibility of polynomials over $\mathbb{F}_q$.

Lemma 2.1. Assume that $n \ge 2$. For any $a \in \mathbb{F}_q$ with Ord$(a) = k$, the binomial $x^n - a$ is irreducible over $\mathbb{F}_q$ if and only if both the following conditions are satisfied :

1. Every prime divisor of n divides k, but does not divide $\frac{q-1}{k}$;
2. If $4/n, \; then \; 4/(q-1)$.

Lemma 2.2. Let $\alpha \in \mathbb{F}_q be \; a \; root \; of \; x^n - 1, where$ $\gcd(q, n) = 1. Then$

$$\sum_{i=0}^{n-1} \alpha^i = \begin{cases} 0 \; if \; \alpha \neq 1 \\ n \; if \alpha = 1. \end{cases}$$

Lemma 2.3. Let l be a positive integer, $P(X) \in \mathbb{F}_q[X]$ be an irreducible polynomial over $\mathbb{F}_q$ of degree $n > 0$. Suppose that $P(0) \neq 0$ and $P(X)$ is of period d,which is equal to the order of any root of $P(X)$.Then $P(X^l)$ is irreducible over $\mathbb{F}_q$, if only if the following conditions three conditions satisfied:

1. Each prime divisor of l divides d; 2. $gcd\left(l,\frac{q^n-1}{d}\right)=1$ and if $4/l$, then $4/(q^n-1)$.

3. Primitive idempotent in $\mathbb{F}_q[x]/(x^{32p^n}-1)$: Let p be an odd prime and $\mathbb{F}_q$ be a finite field with q elements, where $q=8k+3$ for some k and $p^l \parallel q-1$ for integer $l>0$ and $4 \nmid q-1$. In this paper we intend to find primitive idempotents in the ring $\mathbb{F}_q[x]/(x^{32p^n}-1)$ for two different cases $n\le l$ and $n\ge l$. We have an irreducible factorization of $x^{32}-1$ over $\mathbb{F}_q$ as follows:

$$x^{32}-1=(x\pm1)(x^2+1)\left(x^2\pm\sqrt{-2}x-1\right)\left(x^2\pm\sqrt{-(2-\sqrt{2})}x-1\right)\left(x^2\pm\sqrt{-(2+\sqrt{2})}x-1\right)\left(x^2\pm\sqrt{-(2-\sqrt{2+\sqrt{2}})}x-1\right)\left(x^2\pm\sqrt{-(2+\sqrt{2+\sqrt{2}})}x-1\right)\left(x^2\pm\sqrt{-(2-\sqrt{2-\sqrt{2}})}x-1\right)\left(x^2\pm\sqrt{-(2+\sqrt{2-\sqrt{2}})}x-1\right).$$

3.1.When $\boldsymbol{n\le l}$

Theorem 3.1: When $n\le l$, then there are $17p^n$ primitive idempotents in $\mathbb{F}_q[x]/\langle x^{32p^n}-1\rangle$ given as follows:

1. $\phi_{m+k}(x)=\frac{1}{32p^n}\sum_{j=0}^{32p^n-1}(\alpha^k)^j x^j$; $m=0$
2. $\phi_{m+k}(x)=\frac{1}{32p^n}\sum_{j=0}^{32p^n-1}(-\alpha^k)^j x^j$; $m=p^n$
3. $\phi_{m+k}(x)=\frac{1}{16p^n}\sum_{j=0}^{16p^n-1}(-\alpha^{2k})^j x^{2j}$; $m=2p^n$
4. $\phi_{m+k}(x)=\frac{1}{8p^n}\sum_{j=0}^{8p^n-1}(-1)^j\left(\alpha^{4jk}x^{4j}+a\frac{\sqrt{-2}}{2}\alpha^{(4j+3)k}x^{4j+1}(1-x^2)\right)$;

$where\ a=1\ if\ m=4p^n$ and $a=-1\ if\ m=6p^n$

5. $\phi_{m+k}(x)=\frac{1}{16p^n}\sum_{j=0}^{4p^n-1}(-1)^j\left\{\frac{\sqrt{2}}{2}(\sqrt{2}-1-x^4)\alpha^{8jk}x^{8j}+a\frac{\sqrt{2}\sqrt{-(2-\sqrt{2})}}{2}\alpha^{(8j+5)k}x^{8j+1}(1-x^4)-\frac{\sqrt{2}}{2}(\sqrt{2}-1+x^4)\alpha^{(8j+2)k}x^{8j+2}+b\sqrt{-(2-\sqrt{2})}\alpha^{(8j+3)k}x^{8j+7}\right\}$; $where\ a=1,b=-1$ if $m=8p^n$ and $a=-1,b=1$ if $m=10p^n$.

6. $\phi_{m+k}(x)=\frac{1}{16p^n}\sum_{j=0}^{4p^n-1}(-1)^j\left\{\frac{\sqrt{2}}{2}(\sqrt{2}+1+x^4)\alpha^{8jk}x^{8j}+a\frac{\sqrt{2}\sqrt{-(2+\sqrt{2})}}{2}\alpha^{(8j+5)k}x^{8j+1}(1-x^4)-\frac{\sqrt{2}}{2}(\sqrt{2}+1+x^4)\alpha^{(8j+2)k}x^{8j+2}+b\sqrt{-(2+\sqrt{2})}\alpha^{(8j+3)k}x^{8j+7}\right\}$; $where\ a=-1,b=-1$ if $m=12p^n$ and $a=1,b=1$ if $m=14p^n$.

7. $\phi_{m+k}(x)=\frac{1}{16p^n}\sum_{j=0}^{2p^n-1}(-1)^j\left\{-\frac{\sqrt{2}}{2}\left[\sqrt{2+\sqrt{2}}-1-\sqrt{2}+\left(\sqrt{2+\sqrt{2}}-1\right)x^8\right]\alpha^{16jk}x^{16j}+a\frac{\sqrt{2}\sqrt{2+\sqrt{2}}\sqrt{-(2-\sqrt{2+\sqrt{2}})}}{2}(1-x^8)\alpha^{(16j+1)k}x^{16j+1}+\frac{\sqrt{2}}{2}\left[\sqrt{2+\sqrt{2}}-1-\sqrt{2}-\left(\sqrt{2+\sqrt{2}}-1\right)x^8\right]\alpha^{(16j+2)k}x^{16j+2}+b\frac{\sqrt{2}}{2}\sqrt{-(2-\sqrt{2+\sqrt{2}})}(\sqrt{2}+1-x^8)\alpha^{(16j+3)k}x^{16j+3}-\frac{\sqrt{2}}{2}\sqrt{-(2-\sqrt{2+\sqrt{2}})}(\sqrt{2}+1-x^8)\alpha^{(16j+4)k}x^{16j+4}-\frac{\sqrt{2}}{2}\left[1+\left(\sqrt{2}\left(\sqrt{2+\sqrt{2}}-1\right)-1\right)x^8\right]\alpha^{(16j+6)k}x^{16j+6}+c\frac{\sqrt{2}}{2}\sqrt{-\left(2-\sqrt{2+\sqrt{2}}\right)}\left(1-(\sqrt{2}+1)x^8\right)\alpha^{(16j+7)k}x^{16j+7}+d\sqrt{2+\sqrt{2}}\sqrt{-\left(2-\sqrt{2+\sqrt{2}}\right)}\alpha^{(16j+5)k}x^{16j+13}\right\}$;where $a=1,b=1,c=1,d=-1$ if $m=16p^n$ and $a=-1,b=-1,c=-1,d=1$ if $m=18p^n$.

8. $\phi_{m+k}(x)=\frac{1}{16p^n}\sum_{j=0}^{2p^n-1}(-1)^j\left\{\frac{\sqrt{2}}{2}\left(\sqrt{2+\sqrt{2}}+1+\sqrt{2}-\left(\sqrt{2+\sqrt{2}}+1\right)x^8\right)\alpha^{16jk}x^{16j}+a\frac{\sqrt{2}}{2}\sqrt{2+\sqrt{2}}\sqrt{-(2+\sqrt{2+\sqrt{2}})}(1-x^8)\alpha^{(16j+1)k}x^{16j+1}-\frac{\sqrt{2}}{2}\left(\sqrt{2+\sqrt{2}}+1+\sqrt{2}-\left(\sqrt{2+\sqrt{2}}+1\right)x^8\right)\alpha^{(16j+2)k}x^{16j+2}+b\frac{\sqrt{2}}{2}\sqrt{-(2+\sqrt{2+\sqrt{2}})}(\sqrt{2}+1-x^8)\alpha^{(16j+3)k}x^{16j+3}+\frac{\sqrt{2}}{2}\left(\left(\sqrt{2+\sqrt{2}}+1\right)\sqrt{2}+1-x^8\right)\alpha^{(16j+4)k}x^{16j+4}-\frac{\sqrt{2}}{2}\left(\sqrt{2+\sqrt{2}}+\right.$

$1+\sqrt{2}-\left(\sqrt{2+\sqrt{2}}+1\right)x^8\Big)\alpha^{(16j+6)k}x^{16j+6}+c\frac{\sqrt{2}}{2}\sqrt{-\left(2+\sqrt{2+\sqrt{2}}\right)}\left(1-(\sqrt{2}+1)x^8\right)\alpha^{(16j+7)k}x^{16j+7}+$
$d\sqrt{2+\sqrt{2}}\sqrt{-\left(2+\sqrt{2+\sqrt{2}}\right)}\alpha^{(16j+5)k}x^{16j+13}\Big\}$; where $a=-1, b=1, c=1, d=1$ if $m=20p^n$ and $a=1, b=-1, c=-1, d=-1$ if $m=22p^n$

9. $\phi_{m+k}(x)=\frac{1}{16p^n}\sum_{j=0}^{2p^n-1}(-1)^j\Big\{\frac{\sqrt{2}}{2}\left(\sqrt{2-\sqrt{2}}-1+\sqrt{2}+\left(\sqrt{2-\sqrt{2}}-1\right)x^8\right)\alpha^{16jk}x^{16j}+$
$a\frac{\sqrt{2}}{2}\sqrt{2-\sqrt{2}}\sqrt{-(2-\sqrt{2-\sqrt{2}})}(1-x^8)\alpha^{(16j+1)k}x^{16j+1}-\frac{\sqrt{2}}{2}\left(\sqrt{2-\sqrt{2}}-1+\sqrt{2}-\left(\sqrt{2-\sqrt{2}}-1\right)x^8\right)\alpha^{(16j+2)k}x^{16j+2}+$
$b\frac{\sqrt{2}}{2}\sqrt{-(2-\sqrt{2-\sqrt{2}})}(\sqrt{2}-1+x^8)\alpha^{(16j+3)k}x^{16j+3}-\frac{\sqrt{2}}{2}\left(\left(\sqrt{2-\sqrt{2}}-1\right)\sqrt{2}+1+x^8\right)\alpha^{(16j+4)k}x^{16j+4}+\frac{\sqrt{2}}{2}\Big(1-$
$\left(\sqrt{2}\left(\sqrt{2-\sqrt{2}}-1\right)+1\right)x^8\Big)\alpha^{(16j+6)k}x^{16j+6}+c\frac{\sqrt{2}}{2}\sqrt{-\left(2-\sqrt{2-\sqrt{2}}\right)}\left(1-(1-\sqrt{2})x^8\right)\alpha^{(16j+7)k}x^{16j+7}+$
$d\sqrt{2-\sqrt{2}}\sqrt{-\left(2-\sqrt{2-\sqrt{2}}\right)}\alpha^{(16j+5)k}x^{16j+13}\Big\}$; where $a=-1, b=1, c=1, d=1$ if $m=24p^n$ and $a=1, b=-1, c=-1, d=-1$ if $m=26p^n$

10. $\phi_{m+k}(x)=\frac{1}{16p^n}\sum_{j=0}^{2p^n-1}(-1)^j\Big\{-\frac{\sqrt{2}}{2}\left(\sqrt{2-\sqrt{2}}+1-\sqrt{2}+\left(\sqrt{2-\sqrt{2}}+1\right)x^8\right)\alpha^{16jk}x^{16j}+$
$a\frac{\sqrt{2}}{2}\sqrt{2-\sqrt{2}}\sqrt{-(2+\sqrt{2-\sqrt{2}})}(1-x^8)\alpha^{(16j+1)k}x^{16j+1}+\frac{\sqrt{2}}{2}\left(\sqrt{2-\sqrt{2}}+1-\sqrt{2}-\left(\sqrt{2-\sqrt{2}}+1\right)x^8\right)\alpha^{(16j+2)k}x^{16j+2}+$
$b\frac{\sqrt{2}}{2}\sqrt{-(2+\sqrt{2-\sqrt{2}})}(\sqrt{2}-1+x^8)\alpha^{(16j+3)k}x^{16j+3}+\frac{\sqrt{2}}{2}\left(\left(\sqrt{2-\sqrt{2}}+1\right)\sqrt{2}-1+x^8\right)\alpha^{(16j+4)k}x^{16j+4}+\frac{\sqrt{2}}{2}\Big(1+$
$\left(\sqrt{2}\left(\sqrt{2-\sqrt{2}}+1\right)-1\right)x^8\Big)\alpha^{(16j+6)k}x^{16j+6}+c\frac{\sqrt{2}}{2}\sqrt{-\left(2+\sqrt{2-\sqrt{2}}\right)}\left(1-(1-\sqrt{2})x^8\right)\alpha^{(16j+7)k}x^{16j+7}+$
$d\sqrt{2-\sqrt{2}}\sqrt{-\left(2+\sqrt{2-\sqrt{2}}\right)}\alpha^{(16j+5)k}x^{16j+13}\Big\}$; where $a=1, b=1, c=-1, d=1$ if $m=28p^n$ and $a=-1, b=-1, c=1, d=-1$ if $m=30p^n$ and $k=0,1,\dots,p^n-1$.

Proof. If $n\le l$, then the irreducible factorization of $x^{32p^n}-1$ over $\mathbb{F}_q$ is as follows:

$x^{32p^n}-1=\prod_{k=0}^{p^n-1}(x\pm\alpha^{-k})(x^2+\alpha^{-2k})(x^2\pm\sqrt{-2}\alpha^{-k}x-\alpha^{-2k})\left(x^2\pm\sqrt{-(2-\sqrt{2})}\alpha^{-k}x-\alpha^{-2k}\right)\Big(x^2\pm$
$\sqrt{-(2+\sqrt{2})}\alpha^{-k}x-\alpha^{-2k}\Big)\left(x^2\pm\sqrt{-(2-\sqrt{2+\sqrt{2}})}\alpha^{-k}x-\alpha^{-2k}\right)\left(x^2\pm\sqrt{-(2+\sqrt{2+\sqrt{2}})}\alpha^{-k}x-\alpha^{-2k}\right)\Big(x^2\pm$
$\sqrt{-(2-\sqrt{2-\sqrt{2}})}\alpha^{-k}x-\alpha^{-2k}\Big)\left(x^2\pm\sqrt{-(2+\sqrt{2-\sqrt{2}})}\,\alpha^{-k}x-\alpha^{-2k}\right)$, where α^{-1} is a p^{n-th} primitive root of unity.

Now by Chinese Remainder Theorem we define a natural $\mathbb{F}_q$- algebra isomorphism ψ_1 as:

$$\psi_1:\ \mathbb{F}_q[x]/\langle x^{32p^n}-1\rangle\longrightarrow\prod_{k=0}^{p^n-1}\left(\mathcal{R}_k^{(1)}\times\mathcal{R}_k^{(2)}\times\mathcal{R}_k^{(3)}\times\dots\times\mathcal{R}_k^{17)}\right)$$

$$\sum_{j=0}^{32p^n-1}u_jx^j\longrightarrow\left(\prod_{k=0}^{p^n-1}r_k^{(1)},\prod_{k=0}^{p^n-1}r_k^{(2)},\prod_{k=0}^{p^n-1}r_k^{(3)},\dots,\prod_{k=0}^{p^n-1}r_k^{(17)}\right)$$

Where $\mathcal{R}_k^{(1)}=\mathbb{F}_q[x]/\langle x-\alpha^{-k}\rangle$, $\mathcal{R}_k^{(2)}=\mathbb{F}_q[x]/\langle x+\alpha^{-k}\rangle$, $\mathcal{R}_k^{(3)}=\mathbb{F}_q[x]/\langle x^2+\alpha^{-2k}\rangle$, $\mathcal{R}_k^{(4)}=\mathbb{F}_q[x]/\langle x^2-\sqrt{-2}\,\alpha^{-k}x-\alpha^{-2k}\rangle$, $\mathcal{R}_k^{(5)}=\mathbb{F}_q[x]/\langle x^2+\sqrt{-2}\,\alpha^{-k}x-\alpha^{-2k}\rangle$, $\mathcal{R}_k^{(6)}=\mathbb{F}_q[x]/\langle x^2-\sqrt{-(2-\sqrt{2})}\,\alpha^{-k}x-\alpha^{-2k}\rangle$,
$\mathcal{R}_k^{(7)}=\mathbb{F}_q[x]/\langle x^2+\sqrt{-(2-\sqrt{2})}\,\alpha^{-k}x-\alpha^{-2k}\rangle$,

$$\mathcal{R}_k^{(8)}=\mathbb{F}_q[x]/\langle x^2-\sqrt{-(2+\sqrt{2})}\,\alpha^{-k}x-\alpha^{-2k}\rangle,$$

$$\mathcal{R}_k^{(9)}=\mathbb{F}_q[x]/\langle x^2+\sqrt{-(2+\sqrt{2})}\alpha^{-k}x-\alpha^{-2k}\rangle,$$

$$\mathcal{R}_k^{(17)} = \mathbb{F}_q[x]/\langle x^2 + \sqrt{-(2+\sqrt{2-\sqrt{2}})}\,\alpha^{-k}x - \alpha^{-2k}\rangle$$

Also,

$r_k^{(1)} = \sum_{j=0}^{32p^n-1} u_j\,(\alpha^{-k})^j,\ r_k^{(2)} = \sum_{j=0}^{32p^n-1} u_j\,(-\alpha^{-k})^j,\ r_k^{(3)} = \sum_{j=0}^{16p^n-1} u_{2j}\,(-\alpha^{-2k})^j + \sum_{j=0}^{16p^n-1} u_{2j+1}\,(-\alpha^{-2k})^j x,$ $r_k^{(4)} = \sum_{j=0}^{32p^n-1} u_j\left(a_0^{(j,k)} + a_1^{(j,k)}x\right),\ r_k^{(5)} = \sum_{j=0}^{32p^n-1} u_j\left(b_0^{(j,k)} + b_1^{(j,k)}x\right)$, $r_k^{(6)} = \sum_{j=0}^{32p^n-1} u_j\left(a_0^{(j,k)} + a_1^{(j,k)}x\right),$ $r_k^{(7)} = \sum_{j=0}^{32p^n-1} u_j\left(b_0^{(j,k)} + b_1^{(j,k)}x\right),$ $r_k^{(8)} = \sum_{j=0}^{32p^n-1} u_j\left(c_0^{(j,k)} + c_1^{(j,k)}x\right),$ $r_k^{(9)} = \sum_{j=0}^{32p^n-1} u_j\left(d_0^{(j,k)} + d_1^{(j,k)}x\right), r_k^{(10)} = \sum_{j=0}^{32p^n-1} u_j\left(a_0^{(j,k)} + a_1^{(j,k)}x\right), \dots\dots\dots, r_k^{(17)} = \sum_{j=0}^{32p^n-1} u_j\left(t_0^{(j,k)} + t_1^{(j,k)}x\right)$

Where $a_i^{(j,k)}, b_i^{(j,k)}$ for $r_k^{(4)}$ and $r_k^{(5)}$ are defined in lemma 5.1 and 5.2 [7] and $a_i^{(j,k)}, b_i^{(j,k)}, c_i^{(j,k)}, d_i^{(j,k)}$ for $r_k^{(6)}\ to\ r_k^{(9)}$ are defined in lemmas 3.1 to 3.4 for the case $16p^n$ and $a_i^{(j,k)}, b_i^{(j,k)}, c_i^{(j,k)}, d_i^{(j,k)}, e_i^{(j,k)}, m_i^{(j,k)}, n_i^{(j,k)}, t_i^{(j,k)}$ for $r_k^{(10)}$ to $r_k^{(17)}$ are defined in Lemmas 3.1, 3.2,… , 3.8. [5], where $i = 0,1$.

Now a linear space isomorphism ψ_2 is : $\psi_2 : \prod_{k=0}^{p^n-1}(\mathcal{R}_k^1 \times \mathcal{R}_k^2 \times \dots \times \mathcal{R}_k^{17}) \longrightarrow \mathbb{F}_q^{32p^n}$ defined as

$$(B, B', C+C'x, E+E'x, F+F'x, G+G'x, H+H'x, J+J'x, K+K'x, O+O'x, R+R'x, S+S'x, U+U'x, V+V'x, W+W'x, Y+Y'x, Z+Z'x) \longrightarrow$$
$$(B, B', C, C', E, E', F, F', G, G', H, H', J, J', K, K', O, O', R, R', S, S', U, U', V, V', W, W'. Y, Y', Z, Z')$$

Where $B, B', C, C', E, E', F, F', G, G', H, H', J, J', K, K', \dots, Z, Z' \in \mathbb{F}_q^{p^n}$.

Therefore, a linear space isomorphism $\psi = \psi_2 o \psi_1$ is defined as:

$\psi = \psi_2 o \psi_1 \colon \mathbb{F}_q[x]/\langle x^{32p^n} - 1\rangle \longrightarrow \mathbb{F}_q^{32p^n}$, $\phi(x) = \sum_{j=0}^{32p^n-1} u_j\,x^j \longrightarrow \left(u_0, u_1, \dots, u_{32p^n-1}\right) R$

where R is a $32p^n \times 32p^n$ matrix over $\mathbb{F}_q$ defined as $R = \begin{bmatrix} P & Q \\ P & -Q \end{bmatrix}$ … (3.1)

Here P is a $16p^n \times 16p^n$ matrix over $\mathbb{F}_q$ defined in Theorem 3.5 in the paper for primitive idempotents in $\mathbb{F}_q[x]/\langle x^{16p^n} - 1\rangle$ and the matrix Q is defined as follows :

$$Q = \left(Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8, Q_9, Q_{10}, Q_{11}, Q_{12}, Q_{13}, Q_{14}, Q_{15}, Q_{16}\right),$$

is also an $16p^n \times 16p^n$ matrix over $\mathbb{F}_q$ and each Q_i for $i = 1,2,\dots,16$ is a $16p^n \times p^n$ matrix over $\mathbb{F}_q$, which are given as follows :

We take $X_1 = \sqrt{-\left(2 + k\sqrt{2+\sqrt{2}}\right)}, X_2 = \sqrt{-\left(2 + k\sqrt{2-\sqrt{2}}\right)}, Y_1 = \sqrt{2+\sqrt{2}},, Y_2 = \sqrt{2-\sqrt{2}}$ in each of the following matrices.

$$\{Q_t\}_j^k = \begin{pmatrix}
(-1)^u\alpha^{-16u} & \cdots & (-1)^u\alpha^{-16u(p^n-1)} \\
0 & \cdots & 0 \\
(-1)^u\alpha^{-(16u+2)0} & & (-1)^u\alpha^{-(16u+2)(p^n-1)} \\
(-1)^{u+j}X_1\alpha^{-(16u+3)0} & & (-1)^{u+j}X_1\alpha^{-(16u+3)(p^n-1)} \\
(-1)^{u+1}(1+k,Y_1)\alpha^{-(16u+4)0} & & (-1)^{u+1}(1+k,Y_1)\alpha^{-(16u+4)(p^n-1)} \\
(-1)^{u+j+1}k,Y_1X_1\alpha^{-(16u+5)0} & \cdots & (-1)^{u+j+1}k,Y_1X_1\alpha^{-(16u+5)(p^n-1)} \\
(-1)^u\left(1+\sqrt{2}+k,Y_1\right)\alpha^{-(16u+6)0} & \vdots & (-1)^u\left(1+\sqrt{2}+k,Y_1\right)\alpha^{-(16u+6)(p^n-1)} \\
(-1)^{u+j}\left(1+\sqrt{2}\right)X_1\alpha^{-(16u+7)0} & \vdots & (-1)^{u+j}\left(1+\sqrt{2}\right)X_1\alpha^{-(16u+7)(p^n-1)} \\
(-1)^{u+1}\left(1+\sqrt{2}(1+k,Y_1)\right)\alpha^{-(16u+8)0} & \vdots & (-1)^{u+1}\left(1+\sqrt{2}(1+k,Y_1)\right)\alpha^{-(16u+8)(p^n-1)} \\
(-1)^{u+j+1}k\sqrt{2},Y_1X_1\alpha^{-(16u+9)0} & \vdots & (-1)^{u+j+1}k\sqrt{2},Y_1X_1\alpha^{-(16u+9)(p^n-1)} \\
(-1)^u\left(1+\sqrt{2}(1+kY_1)\right)\alpha^{-(16u+10)0} & \vdots & (-1)^u\left(1+\sqrt{2}(1+kY_1)\right)\alpha^{-(16u+10)(p^n-1)} \\
(-1)^{u+j}\left(1+\sqrt{2}\right)X_1\alpha^{-(16u+11)0} & \vdots & (-1)^{u+j}\left(1+\sqrt{2}+\right)X_1\alpha^{-(16u+11)(p^n-1)} \\
(-1)^{u+1}\left(1+\sqrt{2}+kY_1\right)\alpha^{-(16u+12)0} & \cdots & (-1)^{u+1}\left(1+\sqrt{2}+kY_1\right)\alpha^{-(16u+12)(p^n-1)} \\
(-1)^{u+j+1}kY_1X_1\alpha^{-(16u+13)0} & & (-1)^{u+j+1}kY_1X_1\alpha^{-(16u+13)(p^n-1)} \\
(-1)^u(1+kY_1)\alpha^{-(16u+14)0} & & (-1)^u(1+kY_1)\alpha^{-(16u+14)(p^n-1)} \\
(-1)^{u+j}X_1\alpha^{-(16u+15)0} & & (-1)^{u+j}X_1\alpha^{-(16u+15)(p^n-1)}
\end{pmatrix}$$

t	1	3	5	7
J	0	1	0	1
K	-1	-1	1	1

$$\{Q_t\}_j^k = \begin{pmatrix} 0 & \cdots & 0 \\ (-1)^u\alpha^{-16u.0} & \cdots & (-1)^u\alpha^{-16u(p^n-1)} \\ (-1)^{u+j}X_1\alpha^{-(16u+1)0} & & (-1)^{u+j}X_1\alpha^{-(16u+1)(p^n-1)} \\ (-1)^{u+1}(1+kY_1)\alpha^{-(16u+2)0} & & (-1)^{u+1}(1+kY_1)\alpha^{-(16u+2)(p^n-1)} \\ (-1)^{u+j+1}kY_1X_1\alpha^{-(16u+3)0} & & (-1)^{u+j+1}kY_1X_1\alpha^{-(16u+3)(p^n-1)} \\ (-1)^u\left(1+\sqrt{2}+kY_1\right)\alpha^{-(16u+4)0} & \cdots & (-1)^u\left(1+\sqrt{2}+kY_1\right)\alpha^{-(16u+4)(p^n-1)} \\ (-1)^{u+j}\left(1+\sqrt{2}\right)X_1\alpha^{-(16u+5)0} & \cdots & (-1)^{u+j}\left(1+\sqrt{2}\right)X_1\alpha^{-(16u+5)(p^n-1)} \\ (-1)^{u+1}\left(1+\sqrt{2}(1+kY_1)\right)\alpha^{-(16u+6)0} & \cdots & (-1)^{u+1}\left(1+\sqrt{2}(1+kY_1)\right)\alpha^{-(16u+6)(p^n-1)} \\ (-1)^{u+j+1}k\sqrt{2}Y_1X_1\alpha^{-(16u+7)0} & \cdots & (-1)^{u+j+1}k\sqrt{2}Y_1X_1\alpha^{-(16u+7)(p^n-1)} \\ (-1)^u\left(1+\sqrt{2}(1+kY_1)\right)\alpha^{-(16u+8)0} & \cdots & (-1)^u\left(1+\sqrt{2}(1+kY_1)\right)\alpha^{-(16u+8)(p^n-1)} \\ (-1)^{u+j}\left(1+\sqrt{2}\right)X_1\alpha^{-(16u+9)0} & \cdots & (-1)^{u+j}\left(1+\sqrt{2}\right)X_1\alpha^{-(16u+9)(p^n-1)} \\ (-1)^{u+1}\left(1+\sqrt{2}+kY_1\right)\alpha^{-(16u+10)0} & \cdots & (-1)^{u+1}\left(1+\sqrt{2}+kY_1\right)\alpha^{-(16u+10)(p^n-1)} \\ (-1)^{u+j+1}kY_1X_1\alpha^{-(16u+11)0} & \cdots & (-1)^{u+j+1}Y_1X_1\alpha^{-(16u+11)(p^n-1)} \\ (-1)^u(1+kY_1)\alpha^{-(16u+12)0} & & (-1)^u(1+kY_1)\alpha^{-(16u+12)(p^n-1)} \\ (-1)^{u+j}X_1\alpha^{-(16u+13)0} & & (-1)^{u+j}X_1\alpha^{-(16u+13)(p^n-1)} \\ (-1)^{u+1}\alpha^{-(16u+14)0} & & (-1)^{u+1}\alpha^{-(16u+14)(p^n-1)} \end{pmatrix}$$

where t, j, k varies defined as follows:

t	2	4	6	8
j	0	1	0	1
k	-1	-1	1	1

$$\{Q_t\}_j^k = \begin{pmatrix} (-1)^u\alpha^{-16u} & \cdots & (-1)^u\alpha^{-16u(p^n-1)} \\ 0 & \cdots & 0 \\ (-1)^u\alpha^{-(16u+2)0} & & (-1)^u\alpha^{-(16u+2)(p^n-1)} \\ (-1)^{u+j}X_2\alpha^{-(16u+3)0} & & (-1)^{u+j}X_2\alpha^{-(16u+3)(p^n-1)} \\ (-1)^{u+1}(1+kY_2)\alpha^{-(16u+4)0} & & (-1)^{u+1}(1+kY_2)\alpha^{-(16u+4)(p^n-1)} \\ (-1)^{u+j+1}kY_2X_2\alpha^{-(16u+5)0} & \cdots & (-1)^{u+j+1}kY_2X_2\alpha^{-(16u+5)(p^n-1)} \\ (-1)^u\left(1-\sqrt{2}+kY_2\right)\alpha^{-(16u+6)0} & \cdots & (-1)^u\left(1-\sqrt{2}+kY_2\right)\alpha^{-(16u+6)(p^n-1)} \\ (-1)^{u+j}\left(1-\sqrt{2}\right)X_2\alpha^{-(16u+7)0} & \cdots & (-1)^{u+j}\left(1-\sqrt{2}\right)X_2\alpha^{-(16u+7)(p^n-1)} \\ (-1)^{u+1}\left(1-\sqrt{2}(1+kY_2)\right)\alpha^{-(16u+8)0} & \cdots & (-1)^{u+1}\left(1-\sqrt{2}(1+kY_2)\right)\alpha^{-(16u+8)(p^n-1)} \\ (-1)^{u+j+1}k\sqrt{2}Y_2X_2\alpha^{-(16u+9)0} & \cdots & (-1)^{u+j+1}k\sqrt{2}Y_2X_2\alpha^{-(16u+9)(p^n-1)} \\ (-1)^u\left(1-\sqrt{2}(1+kY_2)\right)\alpha^{-(16u+10)0} & \cdots & (-1)^u\left(1-\sqrt{2}(1+kY_2)\right)\alpha^{-(16u+10)(p^n-1)} \\ (-1)^{u+j}\left(1-\sqrt{2}\right)X_2\alpha^{-(16u+11)0} & \cdots & (-1)^{u+j}\left(1-\sqrt{2}\right)X_2\alpha^{-(16u+11)(p^n-1)} \\ (-1)^{u+1}\left(1-\sqrt{2}+kY_2\right)\alpha^{-(16u+12)0} & \cdots & (-1)^{u+1}\left(1-\sqrt{2}+kY_2\right)\alpha^{-(16u+12)(p^n-1)} \\ (-1)^{u+j+1}kY_2X_2\alpha^{-(16u+13)0} & & (-1)^{u+j+1}Y_2X_2\alpha^{-(16u+13)(p^n-1)} \\ (-1)^u(1+kY_2)\alpha^{-(16u+14)0} & & (-1)^u(1+kY_2)\alpha^{-(16u+14)(p^n-1)} \\ (-1)^{u+j}X_2\alpha^{-(16u+15)0} & & (-1)^{u+j}X_2\alpha^{-(16u+15)(p^n-1)} \end{pmatrix}$$

Where t, j, k varies as below table:

t	9	11	13	15
j	0	1	0	1
k	-1	-1	1	1

$$\{Q_t\}_j^k = \begin{pmatrix} 0 & \cdots & 0 \\ (-1)^u\alpha^{-16u.0} & \cdots & (-1)^u\alpha^{-16u(p^n-1)} \\ (-1)^{u+j}X_2\alpha^{-(16u+1)0} & & (-1)^{u+j}X_2\alpha^{-(16u+1)(p^n-1)} \\ (-1)^{u+1}(1+kY_2)\alpha^{-(16u+2)0} & & (-1)^{u+1}(1+kY_2)\alpha^{-(16u+2)(p^n-1)} \\ (-1)^{u+j+1}kY_2X_2\alpha^{-(16u+3)0} & & (-1)^{u+j+1}kY_2X_2\alpha^{-(16u+3)(p^n-1)} \\ (-1)^u\left(1-\sqrt{2}+kY_2\right)\alpha^{-(16u+4)0} & \cdots & (-1)^u\left(1-\sqrt{2}+kY_2\right)\alpha^{-(16u+4)(p^n-1)} \\ (-1)^{u+j}\left(1-\sqrt{2}\right)X_2\alpha^{-(16u+5)0} & \vdots & (-1)^{u+j}(-1)^{u+j}\left(1-\sqrt{2}\right)X_2\alpha^{-(16u+5)(p^n-1)} \\ (-1)^{u+1}\left(1-\sqrt{2}(1+kY_2)\right)\alpha^{-(16u+6)0} & \vdots & (-1)^{u+1}\left(1-\sqrt{2}(1+kY_2)\right)\alpha^{-(16u+6)(p^n-1)} \\ (-1)^{u+j+1}k\sqrt{2}Y_2X_2\alpha^{-(16u+7)0} & \vdots & (-1)^{u+j+1}k\sqrt{2}Y_2X_2\alpha^{-(16u+7)(p^n-1)} \\ (-1)^u\left(1-\sqrt{2}(1+kY_2)\right)\alpha^{-(16u+8)0} & \vdots & (-1)^u\left(1-\sqrt{2}(1+kY_2)\right)\alpha^{-(16u+8)(p^n-1)} \\ (-1)^{u+j}\left(1-\sqrt{2}\right)X_2\alpha^{-(16u+9)0} & \vdots & (-1)^{u+j}\left(1-\sqrt{2}\right)X_2\alpha^{-(16u+9)(p^n-1)} \\ (-1)^{u+1}\left(1-\sqrt{2}+kY_2\right)\alpha^{-(16u+10)0} & \vdots & (-1)^{u+1}\left(1-\sqrt{2}+kY_2\right)\alpha^{-(16u+10)(p^n-1)} \\ (-1)^{u+j+1}kY_2X_2\alpha^{-(16u+11)0} & \cdots & (-1)^{u+j+1}kY_2\alpha^{-(16u+11)(p^n-1)} \\ (-1)^u(1+kY_2)\alpha^{-(16u+12)0} & & (-1)^u(1+kY_2)\alpha^{-(16u+12)(p^n-1)} \\ (-1)^{u+j}X_2\alpha^{-(16u+13)0} & & (-1)^{u+j}X_2\alpha^{-(16u+13)(p^n-1)} \\ (-1)^{u+1}\alpha^{-(16u+14)0} & & (-1)^{u+1}\alpha^{-(16u+14)(p^n-1)} \end{pmatrix}$$

Where t, j, k varies as below table:

t	10	12	14	16
j	0	1	0	1
k	-1	-1	1	1

Set a matrix $T = \frac{1}{8p^n}\begin{bmatrix} T_1 \\ \vdots \\ T_{16} \end{bmatrix}$ of order $16p^n \times 16p^n$ over $\mathbb{F}_q$, where $T_1, T_2, \ldots, T_{16}$ are matrices of order $p^n \times 16p^n$ and transpose T_i' of each matrix T_i for $i = 1, 2, \ldots, 16$ is given as follows :

$$\{T'_t\}_j^k = \begin{pmatrix} (-1)^u\alpha^{-16u} & \cdots & (-1)^u\alpha^{-16u(p^n-1)} \\ 0 & \cdots & 0 \\ (-1)^{u+1}\alpha^{-(16u+2)0} & & (-1)^{u+1}\alpha^{-(16u+2)(p^n-1)} \\ (-1)^{u+j}X_1\alpha^{-(16u+3)0} & & (-1)^{u+j}X_1\alpha^{-(16u+3)(p^n-1)} \\ (-1)^u(1+kY_1)\alpha^{-(16u+4)0} & & (-1)^{u+1}(1+kY_1)\alpha^{-(16u+4)(p^n-1)} \\ (-1)^{u+j+1}kY_1X_1\alpha^{-(16u+5)0} & \cdots & (-1)^{u+j+1}kY_1X_1\alpha^{-(16u+5)(p^n-1)} \\ (-1)^{u+1}\left(1+\sqrt{2}+kY_1\right)\alpha^{-(16u+6)0} & \vdots & (-1)^{u+1}\left(1+\sqrt{2}+kY_1\right)\alpha^{-(16u+6)(p^n-1)} \\ (-1)^{u+j}\left(1+\sqrt{2}\right)X_1\alpha^{-(16u+7)0} & \vdots & (-1)^{u+j}\left(1+\sqrt{2}\right)X_1\alpha^{-(16u+7)(p^n-1)} \\ (-1)^u\left(1+\sqrt{2}(1+kY_1)\right)\alpha^{-(16u+8)0} & \vdots & (-1)^u\left(1+\sqrt{2}(1+kY_1)\right)\alpha^{-(16u+8)(p^n-1)} \\ (-1)^{u+j}k\sqrt{2}Y_1X_1\alpha^{-(16u+9)0} & \vdots & (-1)^{u+j}k\sqrt{2}Y_1X_1\alpha^{-(16u+9)(p^n-1)} \\ (-1)^u\left(1+\sqrt{2}(1+kY_1)\right)\alpha^{-(16u+10)0} & \vdots & (-1)^u\left(1+\sqrt{2}(1+kY_1)\right)\alpha^{-(16u+10)(p^n-1)} \\ (-1)^{u+j+1}\left(1+\sqrt{2}\right)X_1\alpha^{-(16u+11)0} & \vdots & (-1)^{u+j+1}\left(1+\sqrt{2}\right)X_1\alpha^{-(16u+11)(p^n-1)} \\ (-1)^{u+1}\left(1+\sqrt{2}+kY_1\right)\alpha^{-(16u+12)0} & \cdots & (-1)^{u+1}\left(1+\sqrt{2}+kY_1\right)\alpha^{-(16u+12)(p^n-1)} \\ (-1)^{u+j}kY_1X_1\alpha^{-(16u+13)0} & & (-1)^{u+j}kY_1X_1\alpha^{-(16u+13)(p^n-1)} \\ (-1)^u(1+kY_1)\alpha^{-(16u+14)0} & & (-1)^u(1+kY_1)\alpha^{-(16u+14)(p^n-1)} \\ (-1)^{u+j+1}X_1\alpha^{-(16u+15)0} & & (-1)^{u+j+1}X_1\alpha^{-(16u+15)(p^n-1)} \end{pmatrix}$$

Where t, j, k varies as below table:

t	1	3	5	7
j	0	1	0	1
k	-1	-1	1	1

$$\{T'_t\}_j^k = \begin{pmatrix} 0 & \cdots & 0 \\ (-1)^u \alpha^{-16u.0} & \cdots & (-1)^u \alpha^{-16u(p^n-1)} \\ (-1)^{u+j+1} X_1 \alpha^{-(16u+1)0} & \cdots & (-1)^{u+j+1} X_1 \alpha^{-(16u+1)(p^n-1)} \\ (-1)^{u+1}(1+kY_1)\alpha^{-(16u+2)0} & \cdots & (-1)^{u+1}(1+kY_1)\alpha^{-(16u+2)(p^n-1)} \\ (-1)^{u+j} kY_1 X_1 \alpha^{-(16u+3)0} & \cdots & (-1)^{u+j} kY_1 X_1 \alpha^{-(16u+3)(p^n-1)} \\ (-1)^u\left(1+\sqrt{2}+kY_1\right)\alpha^{-(16u+4)0} & \cdots & (-1)^u\left(1+\sqrt{2}+kY_1\right)\alpha^{-(16u+4)(p^n-1)} \\ (-1)^{u+j+1}\left(1+\sqrt{2}\right)X_1\alpha^{-(16u+5)0} & \cdots & (-1)^{u+j+1}\left(1+\sqrt{2}\right)X_1\alpha^{-(16u+5)(p^n-1)} \\ (-1)^{u+1}\left(1+\sqrt{2}(1+kY_1)\right)\alpha^{-(16u+6)0} & \cdots & (-1)^{u+1}\left(1+\sqrt{2}(1+kY_1)\right)\alpha^{-(16u+6)(p^n-1)} \\ (-1)^{u+j} k\sqrt{2} Y_1 X_1 \alpha^{-(16u+7)0} & \cdots & (-1)^{u+j} k\sqrt{2} Y_1 X_1 \alpha^{-(16u+7)(p^n-1)} \\ (-1)^{u+1}\left(1+\sqrt{2}(1+kY_1)\right)\alpha^{-(16u+8)0} & \cdots & (-1)^{u+1}\left(1+\sqrt{2}(1+kY_1)\right)\alpha^{-(16u+8)(p^n-1)} \\ (-1)^{u+j}\left(1+\sqrt{2}\right)X_1\alpha^{-(16u+9)0} & \cdots & (-1)^{u+j}\left(1+\sqrt{2}\right)X_1\alpha^{-(16u+9)(p^n-1)} \\ (-1)^u\left(1+\sqrt{2}+kY_1\right)\alpha^{-(16u+10)0} & \cdots & (-1)^u\left(1+\sqrt{2}+kY_1\right)\alpha^{-(16u+10)(p^n-1)} \\ (-1)^{u+j} Y_1 X_1 \alpha^{-(16u+11)0} & \cdots & (-1)^{u+j}\sqrt{2+\sqrt{2}}X_1\alpha^{-(16u+11)(p^n-1)} \\ (-1)^{u+1}(1+kY_1)\alpha^{-(16u+12)0} & & (-1)^{u+1}(1+kY_1)\alpha^{-(16u+12)(p^n-1)} \\ (-1)^{u+j} X_1 \alpha^{-(16u+13)0} & & (-1)^{u+j} X_1 \alpha^{-(16u+13)(p^n-1)} \\ (-1)^u \alpha^{-(16u+14)0} & & (-1)^u \alpha^{-(16u+14)(p^n-1)} \end{pmatrix}$$

Where t, j, k varies as below table:

t	2	4	6	8
j	0	1	0	1
k	-1	-1	1	1

$$\{T'_t\}_j^k = \begin{pmatrix} (-1)^u \alpha^{-16u} & \cdots & (-1)^u \alpha^{-16u(p^n-1)} \\ 0 & \cdots & 0 \\ (-1)^{u+1} \alpha^{-(16u+2)0} & & (-1)^{u+1} \alpha^{-(16u+2)(p^n-1)} \\ (-1)^{u+j} X_2 \alpha^{-(16u+3)0} & & (-1)^{u+j} X_2 \alpha^{-(16u+3)(p^n-1)} \\ (-1)^u (1+kY_2)\alpha^{-(16u+4)0} & & (-1)^u (1+kY_2)\alpha^{-(16u+4)(p^n-1)} \\ (-1)^{u+j+1} kY_2 X_2 \alpha^{-(16u+5)0} & \cdots & (-1)^{u+j+1} kY_2 X_2 \alpha^{-(16u+5)(p^n-1)} \\ (-1)^{u+1}\left(1-\sqrt{2}+kY_2\right)\alpha^{-(16u+6)0} & \cdots & (-1)^{u+1}\left(1-\sqrt{2}+kY_2\right)\alpha^{-(16u+6)(p^n-1)} \\ (-1)^{u+j}\left(1-\sqrt{2}\right)X_2\alpha^{-(16u+7)0} & \cdots & (-1)^{u+j}\left(1-\sqrt{2}\right)X_2\alpha^{-(16u+7)(p^n-1)} \\ (-1)^u\left(1-\sqrt{2}(1+kY_2)\right)\alpha^{-(16u+8)0} & \cdots & (-1)^u\left(1-\sqrt{2}(1+kY_2)\right)\alpha^{-(16u+8)(p^n-1)} \\ (-1)^{u+1}\sqrt{2}Y_2X_2\alpha^{-(16u+9)0} & \cdots & (-1)^{u+1}\sqrt{2}Y_2X_2\alpha^{-(16u+9)(p^n-1)} \\ (-1)^u\left(1-\sqrt{2}(1+kY_2)\right)\alpha^{-(16u+10)0} & \cdots & (-1)^u\left(1-\sqrt{2}(1+kY_2)\right)\alpha^{-(16u+10)(p^n-1)} \\ (-1)^{u+j+1}\left(1-\sqrt{2}\right)X_2\alpha^{-(16u+11)0} & \cdots & (-1)^{u+j+1}\left(1-\sqrt{2}\right)X_2\alpha^{-(16u+11)(p^n-1)} \\ (-1)^{u+1}\left(1-\sqrt{2}+kY_2\right)\alpha^{-(16u+12)0} & \cdots & (-1)^{u+1}\left(1-\sqrt{2}+kY_2\right)\alpha^{-(16u+12)(p^n-1)} \\ (-1)^{u+j} kY_2 X_2 \alpha^{-(16u+13)0} & & (-1)^{u+j} kY_2 X_2 \alpha^{-(16u+13)(p^n-1)} \\ (-1)^u (1+kY_2)\alpha^{-(16u+14)0} & & (-1)^u (1+kY_2)\alpha^{-(16u+14)(p^n-1)} \\ (-1)^{u+j+1} X_2 \alpha^{-(16u+15)0} & & (-1)^{u+j+1} X_2 \alpha^{-(16u+15)(p^n-1)} \end{pmatrix}$$

Where t, j,k varies as below table:

t	9	11	13	15
j	0	1	0	1
k	-1	-1	1	1

$$\{T'_t\}_j^k = \begin{pmatrix} 0 & \cdots & 0 \\ (-1)^u\alpha^{-16u.0} & \cdots & (-1)^u\alpha^{-16u(p^n-1)} \\ (-1)^{u+j+1}X_2\alpha^{-(16u+1)0} & & (-1)^{u+j+1}X_2\alpha^{-(16u+1)(p^n-1)} \\ (-1)^{u+1}(1+kY_2)\alpha^{-(16u+2)0} & & (-1)^{u+1}(1+kY_2)\alpha^{-(16u+2)(p^n-1)} \\ (-1)^{u+j}Y_2X_2\alpha^{-(16u+3)0} & & (-1)^{u+j}Y_2X_2\alpha^{-(16u+3)(p^n-1)} \\ (-1)^u(1-\sqrt{2}+kY_2)\alpha^{-(16u+4)0} & \cdots & (-1)^u(1-\sqrt{2}+kY_2)\alpha^{-(16u+4)(p^n-1)} \\ (-1)^{u+j+1}(1-\sqrt{2})X_2\alpha^{-(16u+5)0} & \cdots & (-1)^{u+j+1}(1-\sqrt{2})X_2\alpha^{-(16u+5)(p^n-1)} \\ (-1)^{u+1}\left(1-\sqrt{2}(1+kY_2)\right)\alpha^{-(16u+6)0} & \cdots & (-1)^{u+1}\left(1-\sqrt{2}(1+kY_2)\right)\alpha^{-(16u+6)(p^n-1)} \\ (-1)^{u+j+1}k\sqrt{2}Y_2X_2\alpha^{-(16u+7)0} & \cdots & (-1)^{u+j+1}k\sqrt{2}Y_2X_2\alpha^{-(16u+7)(p^n-1)} \\ (-1)^{u+1}\left(1-\sqrt{2}(1+kY_2)\right)\alpha^{-(16u+8)0} & \cdots & (-1)^{u+1}\left(1-\sqrt{2}(1+kY_2)\right)\alpha^{-(16u+8)(p^n-1)} \\ (-1)^{u+j}(1-\sqrt{2})X_2\alpha^{-(16u+9)0} & \cdots & (-1)^{u+j}(1-\sqrt{2})X_2\alpha^{-(16u+9)(p^n-1)} \\ (-1)^u(1-\sqrt{2}+kY_2)\alpha^{-(16u+10)0} & \cdots & (-1)^u(1-\sqrt{2}+kY_2)\alpha^{-(16u+10)(p^n-1)} \\ (-1)^{u+j+1}kY_2X_2\alpha^{-(16u+11)0} & \cdots & (-1)^{u+j+1}kY_2X_2\alpha^{-(16u+11)(p^n-1)} \\ (-1)^{u+1}(1+kY_2)\alpha^{-(16u+12)0} & & (-1)^{u+1}(1+kY_2)\alpha^{-(16u+12)(p^n-1)} \\ (-1)^{u+j}X_2\alpha^{-(16u+13)0} & & (-1)^{u+j}\alpha^{-(16u+13)(p^n-1)} \\ (-1)^u\alpha^{-(16u+14)0} & & (-1)^u\alpha^{-(16u+14)(p^n-1)} \end{pmatrix}$$

Where t, j,k varies as below table:

t	10	12	14	16
j	0	1	0	1
k	-1	-1	1	1

Therefore we have $Q\ T = K(say)$ a $16p^n \times 16p^n$ matrix such that $Q\ T = (d_{ij})\ for\ i, j = 0, 1, \dots, 16p^n - 1$,where $d_{ij} = 1$, if $i = j$ and $d_{ij} = 1\ or -1$, if $i \neq j$. To solve it easily, taking n even and $16u \equiv 16u' + 8(mod p^n)$ we have $Q\ T = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$, where A, B are two elementary matrices of order $8p^n \times 8p^n$ as follows:

$$A = \begin{pmatrix} \cdots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \end{pmatrix}, B = \begin{pmatrix} \cdots & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \cdots \end{pmatrix}$$

Now $K^{-1} = \frac{1}{2}\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$. Hence $Q^{-1} = \frac{1}{2}T\begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \frac{1}{8p^n}\begin{bmatrix} M_1 \\ \vdots \\ M_{16} \end{bmatrix}$, where M_i for $i = 1, 2, \dots, 16$

are $p^n \times 16p^n$ matrices. Here transposes M_i' of each matrix M_i, $i = 1, 2, \dots, 16$ are given as follows:

$$\{M_t'\}_j^k = \frac{1}{2}\begin{pmatrix} (-1)^u\left(2+\sqrt{2}(1+kY_1)\right)\alpha^{16u.0} & \cdots & (-1)^u\left(2+\sqrt{2}(1+kY_1)\right)\alpha^{16u(p^n-1)} \\ (-1)^{u+j+1}k\sqrt{2}Y_1X_1\alpha^{(16u+1)0} & \cdots & (-1)^{u+j+1}k\sqrt{2}Y_1X_1\alpha^{(16u+1)(p^n-1)} \\ (-1)^{u+1}\left(2+\sqrt{2}(1+kY_1)\right)\alpha^{(16u+2)0} & \cdots & (-1)^{u+1}\left(2+\sqrt{2}(1+kY_1)\right)\alpha^{(16u+2)(p^n-1)} \\ (-1)^{u+j}(2+\sqrt{2})X_1\alpha^{(16u+3)0} & \cdots & (-1)^{u+j}(2+\sqrt{2})X_1\alpha^{(16u+3)(p^n-1)} \\ (-1)^u\left(\sqrt{2}+2(1+kY_1)\right)\alpha^{(16u+4)0} & \cdots & (-1)^u\left(\sqrt{2}+2(1+kY_1)\right)\alpha^{(16u+4)(p^n-1)} \\ 0 & \cdots & 0 \\ (-1)^{u+1}\sqrt{2}\,\alpha^{(16u+6)0} & \cdots & (-1)^{u+1}\sqrt{2}\,\alpha^{(16u+6)(p^n-1)} \\ (-1)^{u+j}\sqrt{2}X_1\alpha^{(16u+7)0} & \cdots & (-1)^{u+j}\sqrt{2}X_1\alpha^{(16u+7)(p^n-1)} \\ (-1)^u\sqrt{2}(1+kY_1)\,\alpha^{(16u)0} & \cdots & (-1)^u\sqrt{2}(1+kY_1)\,\alpha^{(16u)(p^n-1)} \\ (-1)^{u+j}k\sqrt{2}Y_1X_1\alpha^{(16u+1)0} & \cdots & (-1)^{u+j}k\sqrt{2}Y_1X_1\alpha^{(16u+1)(p^n-1)} \\ (-1)^u\sqrt{2}(1+kY_1)\alpha^{(16u+2)0} & \cdots & (-1)^u\sqrt{2}(1+kY_1)\alpha^{(16u+2)(p^n-1)} \\ (-1)^{u+j+1}\sqrt{2}X_1\alpha^{(16u+3)0} & \cdots & (-1)^{u+j+1}\sqrt{2}X_1\alpha^{(16u+3)(p^n-1)} \\ (-1)^{u+1}\sqrt{2}\alpha^{(16u+4)0} & \cdots & (-1)^{u+1}\sqrt{2}\alpha^{(16u+4)(p^n-1)} \\ (-1)^{u+j}2kY_1X_1\alpha^{(16u+5)0} & \cdots & (-1)^{u+j}k2Y_1X_1\alpha^{(16u+5)(p^n-1)} \\ (-1)^u\left(\sqrt{2}+2(1+kY_1)\right)\alpha^{(16u+6)0} & & (-1)^u\left(\sqrt{2}+2(1+kY_1)\right)\alpha^{(16u+6)(p^n-1)} \\ (-1)^{u+j+1}(2+\sqrt{2})X_1\alpha^{(16u+7)0} & & (-1)^{u+j+1}(2+\sqrt{2})X_1\alpha^{(16u+7)(p^n-1)} \end{pmatrix}$$

Where t, j,k varies as below table:

t	1	3	5	7
j	0	1	0	1
k	-1	-1	1	1

$$\{M_t'\}_j^k = \frac{1}{2}\begin{pmatrix} (-1)^{u+1}k\sqrt{2}Y_1X_1\alpha^{(16u+7)0} & \cdots & (-1)^{u+1}k\sqrt{2}Y_1X_1\alpha^{(16u+7)(p^n-1)} \\ (-1)^u\left(2+\sqrt{2}(1+kY_1)\right)\alpha^{16u.0} & \cdots & (-1)^u\left(2+\sqrt{2}(1+kY_1)\right)\alpha^{16u.(p^n-1)} \\ (-1)^{u+j+1}(2+\sqrt{2})X_1\alpha^{(16u+1)0} & & (-1)^{u+j+1}(2+\sqrt{2})X_1\alpha^{(16u+1)(p^n-1)} \\ (-1)^{u+1}\left(\sqrt{2}+2(1+kY_1)\right)\alpha^{(16u+2)0} & & (-1)^{u+1}\left(\sqrt{2}+2(1+kY_1)\right)\alpha^{(16u+2)(p^n-1)} \\ (-1)^{u+j}k2Y_1X_1\alpha^{(16u+3)0} & \cdots & (-1)^{u+j}k2Y_1X_1\alpha^{(16u+3)(p^n-1)} \\ (-1)^u\sqrt{2}\,\alpha^{(16u+4)0} & \cdots & (-1)^u\sqrt{2}\,\alpha^{(16u+4)(p^n-1)} \\ (-1)^{u+j+1}\sqrt{2}X_1\alpha^{(16u+5)0} & \cdots & (-1)^{u+j+1}\sqrt{2}X_1\alpha^{(16u+5)(p^n-1)} \\ (-1)^{u+1}\sqrt{2}(1+kY_1)\alpha^{(16u+6)0} & \cdots & (-1)^{u+1}\sqrt{2}(1+kY_1)\alpha^{(16u+6)(p^n-1)} \\ (-1)^{u+j}k\sqrt{2}Y_1X_1\alpha^{(16u+7)0} & \cdots & (-1)^{u+j}k\sqrt{2}Y_1X_1\alpha^{(16u+7)(p^n-1)} \\ (-1)^{u+1}\sqrt{2}(1+kY_1)\,\alpha^{16u.0} & \cdots & (-1)^{u+1}\sqrt{2}(1+kY_1)\alpha^{16u.(p^n-1)} \\ (-1)^{u+j}\sqrt{2}X_1\alpha^{(16u+1)0} & \cdots & (-1)^{u+j}\sqrt{2}X_1\alpha^{(16u+1)(p^n-1)} \\ (-1)^u\sqrt{2}\,\alpha^{(16u+2)0} & \cdots & (-1)^u\sqrt{2}\,\alpha^{(16u+2)(p^n-1)} \\ 0 & \cdots & 0 \\ (-1)^{u+1}\left(\sqrt{2}+2(1+kY_1)\right)\alpha^{(16u+4)0} & \cdots & (-1)^{u+1}\left(\sqrt{2}+2(1+kY_1)\right)\alpha^{(16u+4)(p^n-1)} \\ (-1)^{u+j}(2+\sqrt{2})X_1\alpha^{(16u+5)0} & & (-1)^{u+j}(2+\sqrt{2})X_1\alpha^{(16u+5)(p^n-1)} \\ (-1)^u\left(2+\sqrt{2}(1+kY_1)\right)\alpha^{(16u+6)0} & & (-1)^u\left(2+\sqrt{2}(1+kY_1)\right)\alpha^{(16u+6)(p^n-1)} \end{pmatrix}$$

Where t, j,k varies as below table:

t	2	4	6	8
j	0	1	0	1
k	-1	-1	1	1

$$\{M_t'\}_j^k = \frac{1}{2}\begin{pmatrix} (-1)^u\left(2-\sqrt{2}(1+kY_2)\right)\alpha^{16u.0} & \cdots & (-1)^u\left(2-\sqrt{2}(1+kY_2)\right)\alpha^{16u(p^n-1)} \\ (-1)^{u+j}k\sqrt{2}Y_2X_2\alpha^{(16u+1)0} & \cdots & (-1)^{u+j}k\sqrt{2}Y_2X_2\alpha^{(16u+1)(p^n-1)} \\ (-1)^{u+1}\left(2-\sqrt{2}(1+kY_2)\right)\alpha^{(16u+2)0} & & (-1)^{u+1}\left(2-\sqrt{2}(1+kY_2)\right)\alpha^{(16u+2)(p^n-1)} \\ (-1)^{u+j}\left(2-\sqrt{2}\right)X_2\alpha^{(16u+3)0} & & (-1)^{u+j}\left(2-\sqrt{2}\right)X_2\alpha^{(16u+3)(p^n-1)} \\ (-1)^{u+1}\left(\sqrt{2}-2(1+kY_2)\right)\alpha^{(16u+4)0} & \cdots & (-1)^{u+1}\left(\sqrt{2}-2(1+kY_2)\right)\alpha^{(16u+4)(p^n-1)} \\ 0 & \vdots & 0 \\ (-1)^u\sqrt{2}\,\alpha^{(16u+6)0} & \vdots & (-1)^u\sqrt{2}\,\alpha^{(16u+6)(p^n-1)} \\ (-1)^{u+j+1}\sqrt{2}X_2\alpha^{(16u+7)0} & \vdots & (-1)^{u+j+1}\sqrt{2}X_2\alpha^{(16u+7)(p^n-1)} \\ (-1)^{u+1}\sqrt{2}(1+kY_2)\,\alpha^{(16u)0} & \vdots & (-1)^{u+1}\sqrt{2}(1+kY_2)\alpha^{(16u)(p^n-1)} \\ (-1)^{u+j+1}k\sqrt{2}Y_2X_2\alpha^{(16u+1)0} & \vdots & (-1)^{u+j+1}k\sqrt{2}Y_2X_2\alpha^{(16u+1)(p^n-1)} \\ (-1)^{u+1}\sqrt{2}(1+kY_2)\alpha^{(16u+2)0} & \vdots & (-1)^{u+1}\sqrt{2}(1+kY_2)\alpha^{(16u+2)(p^n-1)} \\ (-1)^{u+j}\sqrt{2}X_2\alpha^{(16u+3)0} & \vdots & (-1)^{u+j}\sqrt{2}X_2\alpha^{(16u+3)(p^n-1)} \\ (-1)^u\sqrt{2}\,\alpha^{(16u+4)0} & \vdots & (-1)^u\sqrt{2}\alpha^{(16u+4)(p^n-1)} \\ (-1)^{u+j}k2Y_2X_2\alpha^{(16u+5)0} & \cdots & (-1)^{u+j}k\,2Y_2X_2\alpha^{(16u+5)(p^n-1)} \\ (-1)^{u+1}\left(\sqrt{2}-2(1+kY_2)\right)\alpha^{(16u+6)0} & & (-1)^{u+1}\left(\sqrt{2}-2(1+kY_2)\right)\alpha^{(16u+6)(p^n-1)} \\ (-1)^{u+j+1}\left(2-\sqrt{2}\right)X_2\alpha^{(16u+7)0} & & (-1)^{u+j+1}\left(2-\sqrt{2}\right)X_2\alpha^{(16u+7)(p^n-1)} \end{pmatrix}$$

Where t, j,k varies as below table:

t	9	11	13	15
j	0	1	0	1
k	-1	-1	1	1

$$\{M_t'\}_j^k = \frac{1}{2}\begin{pmatrix} (-1)^{u+j+1}k\sqrt{2}Y_2X_2\alpha^{(16u+7)0} & \cdots & (-1)^{u+j+1}k\sqrt{2}Y_2X_2\alpha^{(16u+7)(p^n-1)} \\ (-1)^u\left(2-\sqrt{2}(1+kY_2)\right)\alpha^{16u.0} & \cdots & (-1)^u\left(2-\sqrt{2}(1+kY_2)\right)\alpha^{16u.(p^n-1)} \\ (-1)^{u+j+1}\left(2-\sqrt{2}\right)X_2\alpha^{(16u+1)0} & & (-1)^{u+j+1}\left(2-\sqrt{2}\right)X_2\alpha^{(16u+1)(p^n-1)} \\ (-1)^u\left(\sqrt{2}-2(1+kY_2)\right)\alpha^{(16u+2)0} & & (-1)^u\left(\sqrt{2}-2(1+kY_2)\right)\alpha^{(16u+2)(p^n-1)} \\ (-1)^{u+j}k2Y_2X_2\alpha^{(16u+3)0} & \cdots & (-1)^{u+j}k2Y_2X_2\alpha^{(16u+3)(p^n-1)} \\ (-1)^{u+1}\sqrt{2}\,\alpha^{(16u+4)0} & \vdots & (-1)^{u+1}\sqrt{2}\,\alpha^{(16u+4)(p^n-1)} \\ (-1)^{u+j}\sqrt{2}X_2\alpha^{(16u+5)0} & \vdots & (-1)^{u+j}\sqrt{2}X_2\alpha^{(16u+5)(p^n-1)} \\ (-1)^u\sqrt{2}(1+kY_2)\alpha^{(16u+6)0} & \vdots & (-1)^u\sqrt{2}(1+kY_2)\alpha^{(16u+6)(p^n-1)} \\ (-1)^{u+j+1}k\sqrt{2}Y_2X_2\alpha^{(16u+7)0} & \vdots & (-1)^{u+j+1}k\sqrt{2}Y_2X_2\alpha^{(16u+7)(p^n-1)} \\ (-1)^u\sqrt{2}(1+kY_2)\alpha^{16u.0} & \vdots & (-1)^u\sqrt{2}(1+kY_2)\alpha^{16u.(p^n-1)} \\ (-1)^{u+j+1}\sqrt{2}X_2\alpha^{(16u+1)0} & \vdots & (-1)^{u+j+1}\sqrt{2}X_2\alpha^{(16u+1)(p^n-1)} \\ (-1)^{u+1}\sqrt{2}\,\alpha^{(16u+2)0} & \vdots & (-1)^{u+1}\sqrt{2}\,\alpha^{(16u+2)(p^n-1)} \\ 0 & \vdots & 0 \\ (-1)^u\left(\sqrt{2}-2(1+kY_2)\right)\alpha^{(16u+4)0} & \cdots & (-1)^u\left(\sqrt{2}-2(1+kY_2)\right)\alpha^{(16u+4)(p^n-1)} \\ (-1)^{u+j}\left(2-\sqrt{2}\right)X_2\alpha^{(16u+5)0} & & (-1)^{u+j}\left(2-\sqrt{2}\right)X_2\alpha^{(16u+5)(p^n-1)} \\ (-1)^u\left(2-\sqrt{2}(1+kY_2)\right)\alpha^{(16u+6)0} & & (-1)^u\left(2-\sqrt{2}(1+kY_2)\right)\alpha^{(16u+6)(p^n-1)} \end{pmatrix}$$

Where t, j,k varies as below table:

t	10	12	14	16
j	0	1	0	1
k	-1	-1	1	1

Therefore, from (4.1) we have $R^{-1}=\frac{1}{2}\begin{bmatrix}P^{-1} & P^{-1}\\ S^{-1} & -S^{-1}\end{bmatrix}$. Now using $\mathbb{F}_q$- algebra isomorphism we have the primitive idempotents $\phi_j(x)=\sum_{i=0}^{32p^n-1}a_ix^i$; $j=0,1,2,\ldots,9p^n-1,10p^n,\ldots,15p^n-1,16p^n,\ldots,17p^n-1,\ldots,18p^n,\ldots,19p^n-1,20p^n,\ldots,31p^n-1$ in $\mathbb{F}_q[x]/\langle x^{32P^n}-1\rangle$ as follows:

$$\psi\left(\phi_j(x)\right)=\psi\left(\sum_{i=0}^{32p^n-1}a_ix^i\right)=\left(a_0,a_1,\ldots,a_{32p^n-1}\right)P=e_j$$

$$\left(a_0,a_1,\ldots,a_{32p^n-1}\right)=e_jP^{-1}=\vartheta_{j+1}$$

Where ϑ_{j+1} is j-th row of P^{-1} for $j=0,1,\ldots,17p^n-1$ and e_j are $17p^n$ vectors of $\mathbb{F}_q^{32p^n}$ corresponding to all primitive idempotents of $\prod_{j=0}^{p^n-1}\mathcal{R}_j^1\times\prod_{j=0}^{p^n-1}\mathcal{R}_j^2\times,\ldots,\times\prod_{j=0}^{p^n-1}\mathcal{R}_j^{17}$ such that $e_0=(1,0,\ \ldots\ ,0),e_1=(0,1,0,\ldots\ ,0)$ and so on. Therefore

$\phi_{16p^n+k}(x)=\frac{1}{16p^n}\sum_{j=0}^{2p^n-1}(-1)^j\Bigg\{-\frac{\sqrt{2}}{2}\left(\sqrt{2+\sqrt{2}}-1-\sqrt{2}+\left(\sqrt{2+\sqrt{2}}-1\right)x^8\right)\alpha^{16jk}x^{16j}+\frac{\sqrt{2}\sqrt{2+\sqrt{2}}\sqrt{-(2-\sqrt{2+\sqrt{2}})}}{2}(1-x^8)\alpha^{(16j+1)k}x^{16j+1}+\frac{\sqrt{2}}{2}\left(\sqrt{2+\sqrt{2}}-1-\sqrt{2}-\left(\sqrt{2+\sqrt{2}}-1\right)x^8\right)\alpha^{(16j+2)k}x^{16j+2}+\frac{\sqrt{2}}{2}\sqrt{-(2-\sqrt{2+\sqrt{2}})}(\sqrt{2}+1-x^8)\alpha^{(16j+3)k}x^{16j+3}-\frac{\sqrt{2}}{2}\sqrt{-(2-\sqrt{2+\sqrt{2}})}(\sqrt{2}+1-x^8)\alpha^{(16j+4)k}x^{16j+4}-\frac{\sqrt{2}}{2}\left(1+\left(\sqrt{2}\left(\sqrt{2+\sqrt{2}}-1\right)-1\right)x^8\right)\alpha^{(16j+6)k}x^{16j+6}+\frac{\sqrt{2}}{2}\sqrt{-\left(2-\sqrt{2+\sqrt{2}}\right)}\left(1-(\sqrt{2}+1)x^8\right)\alpha^{(16j+7)k}x^{16j+7}-\sqrt{2+\sqrt{2}}\sqrt{-\left(2-\sqrt{2+\sqrt{2}}\right)}\alpha^{(16j+5)k}x^{16j+13}\Bigg\}$, where $k=0,1,\ldots,p^n-1$. Similarly, we get all the required primitive idempotents in $\mathbb{F}_q[x]/\langle x^{32p^n}-1\rangle$ for irreducible cyclic codes of length $32p^n$.

When $n\geq l$, then $n=vl+s,\ 0\leq s<l$. Let $v=1$ then $n=l+s$. Now using lemma 2.1and 2.2 we have the factorization of $x^{32p^n}-1$ over $\mathbb{F}_q$ is as follows :

$x^{32p^{l+s}}-1=\prod_{k=0}^{p^l-1}\left(x^{p^s}\pm\alpha^{-k}\right)\left(x^{2p^s}+\alpha^{-2k}\right)\left(x^{2p^s}\pm\sqrt{-2}\,\alpha^{-k}x^{p^s}-\alpha^{-2k}\right)\left(x^{2p^s}\pm\sqrt{-(2-\sqrt{2})}\,\alpha^{-k}x^{p^s}-\alpha^{-2k}\right)\left(x^{2p^s}\pm\sqrt{-(2+\sqrt{2})}\,\alpha^{-k}x^{p^s}-\alpha^{-2k}\right)\left(x^{2p^s}\pm\sqrt{-(2-\sqrt{2+\sqrt{2}})}\alpha^{-k}x^{p^s}-\alpha^{-2k}\right)\left(x^{2p^s}\pm\sqrt{-(2+\sqrt{2+\sqrt{2}})}\alpha^{-k}x^{p^s}-\alpha^{-2k}\right)\left(x^{2p^s}\pm\sqrt{-(2-\sqrt{2-\sqrt{2}})}\alpha^{-k}x^{p^s}-\alpha^{-2k}\right)\left(x^{2p^s}\pm\sqrt{-(2+\sqrt{2-\sqrt{2}})}\,\alpha^{-k}x^{p^s}-\alpha^{-2k}\right)$, Where α^{-1} is a p^l-th primitive root of unity over $\mathbb{F}_q$.

Now using lemma 4.1[5] about irreducible factorization of $x^{32p^n}-1$ over $\mathbb{F}_q$ for $n\geq l$, we get the primitive idempotents in the same way as theorem 3.1 which are given as follows:

Theorem 3.2. If $n\geq l$, then $n=vl+s$. Let $v=1$ and $k=p^t.c$ with $\gcd(p,c)=1$. Then all primitive idempotents in $\mathbb{F}_q[x]/\langle x^{16p^{l+s}}-1\rangle$ computed are as follows:

(1) For $t=0$, corresponding primitive idempotents are as follows:

$$\phi_k(x)=\frac{1}{32p^l}\sum_{j=0}^{32p^l-1}(\alpha^k)^jx^{jp^s};\ k=0,1,\ldots,p^n-1$$

$$\phi_{p^n+k}(x)=\frac{1}{32p^l}\sum_{j=0}^{32p^l-1}(-\alpha^k)^jx^{jp^s};k=0,1,\ldots,p^n-1$$

$$\phi_{2p^n+k}(x)=\frac{1}{16p^l}\sum_{j=0}^{16p^l-1}(-\alpha^{2k})^jx^{2jp^s};k=0,1,\ldots,p^n-1$$

$\phi_{m+k}(x)=\frac{1}{16p^l}\sum_{j=0}^{8p^l-1}(-1)^j\left(\alpha^{4jk}x^{4jp^s}+a\frac{\sqrt{-2}}{2}\alpha^{(4j+3)k}x^{(4j+1)p^s}(1-x^2)\right)$; *where* $a=1,-1$ for $m=4p^n$ and $6p^n$ respectively.

$$\phi_{m+k}(x) = \frac{1}{16p^l}\sum_{j=0}^{4p^l-1}(-1)^j\left\{\frac{\sqrt{2}}{2}\left(\sqrt{2}-1-x^4\right)\alpha^{8jk}x^{8jp^s} + a\frac{\sqrt{2}\sqrt{-(2-\sqrt{2})}}{2}\alpha^{(8j+5)k}x^{(8j+1)p^s}(1-x^4) - \frac{\sqrt{2}}{2}\left(\sqrt{2}-1+x^4\right)\alpha^{(8j+2)k}x^{(8j+2)p^s} + b\sqrt{-(2-\sqrt{2})}\alpha^{(8j+3)k}x^{(8j+7)p^s}\right\}$$; where $a = 1, b = -1$ for $m = 8p^n$ and $a = -1, b = 1$ for $m = 10p^n$.

$$\phi_{m+k}(x) = \frac{1}{16p^l}\sum_{j=0}^{4p^l-1}(-1)^j\left\{\frac{\sqrt{2}}{2}\left(\sqrt{2}+1+x^4\right)\alpha^{8jk}x^{8jp^s} + a\frac{\sqrt{2}\sqrt{-(2+\sqrt{2})}}{2}\alpha^{(8j+5)k}x^{(8j+1)p^s}(1-x^4) - \frac{\sqrt{2}}{2}\left(\sqrt{2}+1+x^4\right)\alpha^{(8j+2)k}x^{(8j+2)p^s} + b\sqrt{-(2+\sqrt{2})}\alpha^{(8j+3)k}x^{(8j+7)p^s}\right\}$$; where $a = -1, b = -1$ for $m = 12p^l$ and $a = 1, b = 1$ for $m = 14p^l$.

$$\phi_{16p^l+k}(x) = \frac{1}{16p^l}\sum_{j=0}^{2p^l-1}(-1)^j\left\{-\frac{\sqrt{2}}{2}\left(\sqrt{2+\sqrt{2}}-1-\sqrt{2}+\left(\sqrt{2+\sqrt{2}}-1\right)x^8\right)\alpha^{16jk}x^{16jp^s} + a\frac{\sqrt{2}\sqrt{2+\sqrt{2}}\sqrt{-(2-\sqrt{2+\sqrt{2}})}}{2}(1-x^8)\alpha^{(16j+1)k}x^{(16j+1)p^s} + \frac{\sqrt{2}}{2}\left(\sqrt{2+\sqrt{2}}-1-\sqrt{2}-\left(\sqrt{2+\sqrt{2}}-1\right)x^8\right)\alpha^{(16j+2)k}x^{(16j+2)p^s} + b\frac{\sqrt{2}}{2}\sqrt{-(2-\sqrt{2+\sqrt{2}})}(\sqrt{2}+1-x^8)\alpha^{(16j+3)k}x^{(16j+3)p^s} - \frac{\sqrt{2}}{2}\sqrt{-(2-\sqrt{2+\sqrt{2}})}(\sqrt{2}+1-x^8)\alpha^{(16j+4)k}x^{(16j+4)p^s} - \frac{\sqrt{2}}{2}\left(1+\left(\sqrt{2}\left(\sqrt{2+\sqrt{2}}-1\right)-1\right)x^8\right)\alpha^{(16j+6)k}x^{(16j+6)p^s} + c\frac{\sqrt{2}}{2}\sqrt{-\left(2-\sqrt{2+\sqrt{2}}\right)}\left(1-(\sqrt{2}+1)x^8\right)\alpha^{(16j+7)k}x^{(16j+7)p^s} + d\sqrt{2+\sqrt{2}}\sqrt{-\left(2-\sqrt{2+\sqrt{2}}\right)}\alpha^{(16j+5)k}x^{(16j+13)p^s}\right\},$$

Where $a = 1, b = 1, c = 1, d = -1$ for $m = 16p^l$ and $a = -1, b = -1, c = -1, d = 1$ for $18p^l$

$$\phi_{20p^l+k}(x) = \frac{1}{16p^l}\sum_{j=0}^{2p^l-1}(-1)^j\left\{\frac{\sqrt{2}}{2}\left(\sqrt{2+\sqrt{2}}+1+\sqrt{2}-\left(\sqrt{2+\sqrt{2}}+1\right)x^8\right)\alpha^{16jk}x^{16jp^s} + a\frac{\sqrt{2}}{2}\sqrt{2+\sqrt{2}}\sqrt{-(2+\sqrt{2+\sqrt{2}})}(1-x^8)\alpha^{(16j+1)k}x^{(16j+1)p^s} - \frac{\sqrt{2}}{2}\left(\sqrt{2+\sqrt{2}}+1+\sqrt{2}-\left(\sqrt{2+\sqrt{2}}+1\right)x^8\right)\alpha^{(16j+2)k}x^{(16j+2)p^s} + b\frac{\sqrt{2}}{2}\sqrt{-(2+\sqrt{2+\sqrt{2}})}(\sqrt{2}+1-x^8)\alpha^{(16j+3)k}x^{(16j+3)p^s} + \frac{\sqrt{2}}{2}\left(\left(\sqrt{2+\sqrt{2}}+1\right)\sqrt{2}+1-x^8\right)\alpha^{(16j+4)k}x^{(16j+4)p^s} - \frac{\sqrt{2}}{2}\left(1-\left(\sqrt{2}\left(\sqrt{2+\sqrt{2}}+1\right)+1\right)x^8\right)\alpha^{(16j+6)k}x^{(16j+6)p^s} + c\frac{\sqrt{2}}{2}\sqrt{-\left(2+\sqrt{2+\sqrt{2}}\right)}\left(1-(\sqrt{2}+1)x^8\right)\alpha^{(16j+7)k}x^{(16j+7)p^s} + d\sqrt{2+\sqrt{2}}\sqrt{-\left(2+\sqrt{2+\sqrt{2}}\right)}\alpha^{(16j+5)k}x^{(16j+13)p^s}\right\},$$

Where $a = -1, b = 1, c = 1, d = 1$ for $m = 20p^l$ and $a = 1, b = -1, c = -1, d = -1$ for $22p^l$

$$\phi_{24p^l+k}(x) = \frac{1}{16p^l}\sum_{j=0}^{2p^l-1}(-1)^j\left\{\frac{\sqrt{2}}{2}\left(\sqrt{2-\sqrt{2}}-1+\sqrt{2}+\left(\sqrt{2-\sqrt{2}}-1\right)x^8\right)\alpha^{16jk}x^{16jp^s} + a\frac{\sqrt{2}}{2}\sqrt{2-\sqrt{2}}\sqrt{-(2-\sqrt{2-\sqrt{2}})}(1-x^8)\alpha^{(16j+1)k}x^{(16j+1)p^s} - \frac{\sqrt{2}}{2}\left(\sqrt{2-\sqrt{2}}-1+\sqrt{2}-\left(\sqrt{2-\sqrt{2}}-1\right)x^8\right)\alpha^{(16j+2)k}x^{(16j+2)p^s} + b\frac{\sqrt{2}}{2}\sqrt{-(2-\sqrt{2-\sqrt{2}})}(\sqrt{2}-1+x^8)\alpha^{(16j+3)k}x^{(16j+3)p^s} - \frac{\sqrt{2}}{2}\left(\left(\sqrt{2-\sqrt{2}}-1\right)\sqrt{2}+1+x^8\right)\alpha^{(16j+4)k}x^{(16j+4)p^s} + \frac{\sqrt{2}}{2}\left(1-\left(\sqrt{2}\left(\sqrt{2-\sqrt{2}}-1\right)+1\right)x^8\right)\alpha^{(16j+6)k}x^{(16j+6)p^s} + c\frac{\sqrt{2}}{2}\sqrt{-\left(2-\sqrt{2-\sqrt{2}}\right)}\left(1-(1-\sqrt{2})x^8\right)\alpha^{(16j+7)k}x^{(16j+7)p^s} + d\sqrt{2-\sqrt{2}}\sqrt{-\left(2-\sqrt{2-\sqrt{2}}\right)}\alpha^{(16j+5)k}x^{(16j+13)p^s}\right\},$$

Where $a = -1, b = 1, c = -1, d = -1$ for $m = 24p^l$ and $a = 1, b = -1, c = 1, d = 1$ for $26p^l$

$$\phi_{28p^l+k}(x) = \frac{1}{16p^l}\sum_{j=0}^{2p^l-1}(-1)^j\left\{-\frac{\sqrt{2}}{2}\left(\sqrt{2-\sqrt{2}}+1-\sqrt{2}+\left(\sqrt{2-\sqrt{2}}+1\right)x^8\right)\alpha^{16jk}x^{16jp^s} + \frac{\sqrt{2}}{2}\sqrt{2-\sqrt{2}}\sqrt{-(2+\sqrt{2-\sqrt{2}})}(1-x^8)\alpha^{(16j+1)k}x^{(16j+1)p^s} + \frac{\sqrt{2}}{2}\left(\sqrt{2-\sqrt{2}}+1-\sqrt{2}-\left(\sqrt{2-\sqrt{2}}+1\right)x^8\right)\alpha^{(16j+2)k}x^{(16j+2)p^s} + \frac{\sqrt{2}}{2}\sqrt{-(2+\sqrt{2-\sqrt{2}})}(\sqrt{2}-1+x^8)\alpha^{(16j+3)k}x^{(16j+3)p^s} + \frac{\sqrt{2}}{2}\left(\left(\sqrt{2-\sqrt{2}}+1\right)\sqrt{2}-1+\right.$$

$x^8\Big)\alpha^{(16j+4)k}x^{(16j+4)p^s}+\frac{\sqrt{2}}{2}\Big(1+\Big(\sqrt{2}\Big(\sqrt{2-\sqrt{2}}+1\Big)-1\Big)x^8\Big)\alpha^{(16j+6)k}x^{(16j+6)p^s}-\frac{\sqrt{2}}{2}\sqrt{-\Big(2+\sqrt{2-\sqrt{2}}\Big)}\Big(1-$

$(1-\sqrt{2})x^8\Big)\alpha^{(16j+7)k}x^{(16j+7)p^s}+\sqrt{2-\sqrt{2}}\sqrt{-\Big(2+\sqrt{2-\sqrt{2}}\Big)}\alpha^{(16j+5)k}x^{(16j+13)p^s}\Big\},$

Where $a=1, b=1, c=-1, d=1$ for $m=28p^l$ and $a=-1, b=-1, c=1, d=-1$ for $30p^l$

(2) For $0\le t<s$, by Lemma 3.6. polynomials $x^{p^{s-t}}\pm\xi_{p^t}^{-i}\alpha^{-c}$, $x^{2p^{s-t}}+\xi_{p^t}^{-i}\alpha^{-2c}$, $x^{2p^{s-t}}\pm\sqrt{-2}\,\xi_{p^t}^{-i}\alpha^{-c}x^{p^{s-t}}-\xi_{p^t}^{-2i}\alpha^{-2c}$, $x^{2p^{s-t}}\pm\sqrt{-(2-\sqrt{2})}\,\xi_{p^t}^{-i}\alpha^{-c}x^{p^{s-t}}-\xi_{p^t}^{-2i}\alpha^{-2c}$, , … , $x^{2p^{s-t}}\pm\sqrt{-(2+\sqrt{2-\sqrt{2}})}\,\xi_{p^t}^{-i}\alpha^{-c}x^{p^{s-t}}-\xi_{p^t}^{-2i}\alpha^{-2c}$ are irreducible over $\mathbb{F}_q$ and respective primitive idempotents are as:

1. $\phi_k^{(i)}(x)=\frac{1}{32p^{l+s}}\Big\{\sum_{r_1=0}^{p^t-1}\sum_{r_2=0}^{32p^l-1}\xi_{p^t}^{r_1i}\alpha^{r_1c+kr_2}x^{r_1p^{s-t}+r_2p^s}\Big\}$; where $m=0$

2. $\phi_{p^l+k}^{(i)}(x)=\frac{1}{32p^{l+s}}\Big\{\sum_{r_1=0}^{p^t-1}\sum_{r_2=0}^{32p^l-1}(-1)^{r_1+r_2}\xi_{p^t}^{r_1i}\alpha^{r_1c+kr_2}x^{r_1p^{s-t}+r_2p^s}\Big\}$; where $m=p^l$

3. $\phi_{2p^l+k}^{(i)}(x)=\frac{1}{16p^{l+s}}\Big\{\sum_{r_1=0}^{p^t-1}\sum_{r_2=0}^{16p^l-1}(-1)^{r_1+r_2}\xi_{p^t}^{r_1i}\alpha^{2r_1c+2kr_2}x^{2r_1p^{s-t}+2r_2p^s}\Big\}$; $m=2p^l$

4. $\phi_{m+k}^{(i)}(x)=\frac{1}{16p^{l+t}}\Big\{\sum_{r_1=0}^{2p^l-1}\sum_{r_2=0}^{p^t-1}(-1)^{r_1}\Big(\nu^{4r_1i}x^{4r_1p^{s-t}}+a\frac{\sqrt{-2}}{2}\nu^{(4r_1+3)i}x^{(4r_1+1)p^{s-t}}\big(1-x^{2p^{s-t}}\big)x^{32r_2p^{l+s-t}}\Big)\Big\}$; *where* $a=1,-1$ for $m=4p^l$ and $6p^l$ respectively.

5. $\phi_{m+k}^{(i)}(x)=\frac{1}{16p^{l+t}}\Big\{\sum_{r_1=0}^{2p^l-1}\sum_{r_2=0}^{p^t-1}(-1)^{r_1}\Big(\frac{\sqrt{2}}{2}(\sqrt{2}-1-x^{4p^{s-t}})\nu^{8r_1i}x^{8r_1p^{s-t}}+a\frac{\sqrt{2}}{2}\sqrt{-(2-\sqrt{2})}\nu^{(8r_1+5)i}(1-$
$x^{4p^{s-t}})x^{(8r_1+1)p^{s-t}}-\frac{\sqrt{2}}{2}(\sqrt{2}-1+x^{4p^{s-t}})\nu^{(8r_1+2)i}x^{(8r_1+2)p^{s-t}}+b\sqrt{-(2-\sqrt{2})}\,\nu^{(8r_1+3)i}x^{(8r_1+7)p^{s-t}}\Big)x^{32r_2p^{l+s-t}}\Big\}$; where $a=1$, $b=-1$ for $m=8p^l$ and $a=-1$, $b=1$ for $m=10p^l$

6. $\phi_{m+k}^{(i)}(x)=\frac{1}{16p^{l+t}}\Big\{\sum_{r_1=0}^{2p^l-1}\sum_{r_2=0}^{p^t-1}(-1)^{r_1}\Big(\frac{\sqrt{2}}{2}(\sqrt{2}+1+x^{4p^{s-t}})\nu^{8r_1i}x^{8r_1p^{s-t}}+a\frac{\sqrt{2}}{2}\sqrt{-(2+\sqrt{2})}\nu^{(8r_1+5)i}(1-$
$x^{4p^{s-t}})x^{(8r_1+1)p^{s-t}}-\frac{\sqrt{2}}{2}(\sqrt{2}+1+x^{4p^{s-t}})\nu^{(8r_1+2)i}x^{(8r_1+2)p^{s-t}}+b\sqrt{-(2+\sqrt{2})}\,\nu^{(8r_1+3)i}x^{(8r_1+7)p^{s-t}}\Big)x^{32r_2p^{l+s-t}}\Big\}$; where $a=-1$, $b=-1$ for $m=12p^l$ and $a=1$, $b=1$ for $m=14p^l$

7. $\phi_{m+k}^{(i)}(x)=\frac{1}{16p^{l+t}}\Big\{\sum_{r_1=0}^{2p^l-1}\sum_{r_2=0}^{p^t-1}(-1)^{r_1}\Big(-\frac{\sqrt{2}}{2}\Big(\sqrt{2+\sqrt{2}}-1-\sqrt{2}+\Big(\sqrt{2+\sqrt{2}}-1\Big)x^{8p^{s-t}}\Big)\Big)\nu^{16r_1i}x^{16r_1p^{s-t}}+$
$a\frac{\sqrt{2}\sqrt{2+\sqrt{2}}\sqrt{-(2-\sqrt{2+\sqrt{2}})}}{2}(1-x^{8p^{s-t}})\nu^{(16r_1+1)i}x^{(16r_1+1)p^{s-t}}+\frac{\sqrt{2}}{2}\Big(\sqrt{2+\sqrt{2}}-1-\sqrt{2}-\Big(\sqrt{2+\sqrt{2}}-$
$1\Big)x^{8p^{s-t}}\Big)\alpha^{(16r_1+2)i}x^{(16r_1+2)p^{s-t}}+b\frac{\sqrt{2}}{2}\sqrt{-(2-\sqrt{2+\sqrt{2}})}(\sqrt{2}+1-x^{8p^{s-t}})\nu^{(16r_1+3)i}x^{(16r_1+3)p^{s-t}}-\frac{\sqrt{2}}{2}\Big(\sqrt{2}\Big(\sqrt{2+\sqrt{2}}-1\Big)-$
$1+x^{8p^{s-t}}\Big)\nu^{(16r_1+4)k}x^{(16r_1+4)p^{s-t}}-\frac{\sqrt{2}}{2}\Big(1+\Big(\sqrt{2}\Big(\sqrt{2+\sqrt{2}}-1\Big)-1\Big)x^{8p^{s-t}}\Big)\nu^{(16r_1+6)k}x^{(16r_1+6)p^{s-t}}+$
$c\frac{\sqrt{2}}{2}\sqrt{-\Big(2-\sqrt{2+\sqrt{2}}\Big)}\big(1-(\sqrt{2}+1)x^{8p^{s-t}}\big)\nu^{(16r_1+7)k}x^{(16r_1+7)p^{s-t}}+$
$d\sqrt{2+\sqrt{2}}\sqrt{-\Big(2-\sqrt{2+\sqrt{2}}\Big)}\nu^{(16r_1+5)k}x^{(16r_1+13)p^{s-t}})x^{32r_2p^{l+s-t}}\Big\}$; where $a=1, b=1, c=1, d=-1$ for $m=16p^l$ and $a=-1, b=-1, c=-1, d=1$ for $18p^l$

8. $\phi_{m+k}^{(i)}(x)=\frac{1}{16p^{l+t}}\Big\{\sum_{r_1=0}^{2p^l-1}\sum_{r_2=0}^{p^t-1}(-1)^{r_1}\Big(\frac{\sqrt{2}}{2}\Big(\sqrt{2+\sqrt{2}}+\sqrt{2}+1-\Big(\sqrt{2+\sqrt{2}}+1\Big)x^{8p^{s-t}}\Big)\Big)\nu^{16r_1i}x^{16r_1p^{s-t}}+$
$a\frac{\sqrt{2}\sqrt{2+\sqrt{2}}\sqrt{-(2+\sqrt{2+\sqrt{2}})}}{2}(1-x^{8p^{s-t}})\nu^{(16r_1+1)i}x^{(16r_1+1)p^{s-t}}-\frac{\sqrt{2}}{2}\Big(\sqrt{2+\sqrt{2}}+\sqrt{2}+1-\Big(\sqrt{2+\sqrt{2}}+$
$1\Big)x^{8p^{s-t}}\Big)\alpha^{(16r_1+2)i}x^{(16r_1+2)p^{s-t}}+b\frac{\sqrt{2}}{2}\sqrt{-(2+\sqrt{2+\sqrt{2}})}(\sqrt{2}+1-x^{8p^{s-t}})\nu^{(16r_1+3)i}x^{(16r_1+3)p^{s-t}}+\frac{\sqrt{2}}{2}\Big(\sqrt{2}\Big(\sqrt{2+\sqrt{2}}+1\Big)+$
$1-x^{8p^{s-t}}\Big)\nu^{(16r_1+4)k}x^{(16r_1+4)p^{s-t}}-\frac{\sqrt{2}}{2}\Big(1-\Big(\sqrt{2}\Big(\sqrt{2+\sqrt{2}}+1\Big)+1\Big)x^{8p^{s-t}}\Big)\nu^{(16r_1+6)k}x^{(16r_1+6)p^{s-t}}+$
$c\frac{\sqrt{2}}{2}\sqrt{-\Big(2+\sqrt{2+\sqrt{2}}\Big)}\big(1-(\sqrt{2}+1)x^{8p^{s-t}}\big)\nu^{(16r_1+7)k}x^{(16r_1+7)p^{s-t}}+$

$d\sqrt{2+\sqrt{2}}\sqrt{-\left(2+\sqrt{2+\sqrt{2}}\right)}\,\nu^{(16r_1+5)k}x^{(16r_1+13)p^{s-t}})x^{32r_2p^{l+s-t}}\Bigg\}$; where $a=-1, b=1, c=1, d=1$ for $m=20p^l$ and $a=1, b=-1, c=-1, d=-1$ for $22p^l$

9. $\phi_{m+k}^{(i)}(x)=\frac{1}{16p^{l+t}}\Bigg\{\sum_{r_1=0}^{2p^l-1}\sum_{r_2=0}^{p^t-1}(-1)^{r_1}\left(\frac{\sqrt{2}}{2}\left(\sqrt{2-\sqrt{2}}+\sqrt{2}-1+\left(\sqrt{2-\sqrt{2}}-1\right)x^{8p^{s-t}}\right)\right)\nu^{16r_1i}x^{16r_1p^{s-t}}+$
$a\frac{\sqrt{2}\sqrt{2-\sqrt{2}}\sqrt{-(2-\sqrt{2-\sqrt{2}})}}{2}(1-x^{8p^{s-t}})\nu^{(16r_1+1)i}x^{(16r_1+1)p^{s-t}}-\frac{\sqrt{2}}{2}\Big(\sqrt{2-\sqrt{2}}+\sqrt{2}-1-\Big(\sqrt{2-\sqrt{2}}-$
$1\Big)x^{8p^{s-t}}\Big)\alpha^{(16r_1+2)i}x^{(16r_1+2)p^{s-t}}+b\frac{\sqrt{2}}{2}\sqrt{-(2-\sqrt{2-\sqrt{2}})}(\sqrt{2}-1+x^{8p^{s-t}})\nu^{(16r_1+3)i}x^{(16r_1+3)p^{s-t}}-\frac{\sqrt{2}}{2}\Big(\sqrt{2}\left(\sqrt{2-\sqrt{2}}-1\right)+$
$1+x^{8p^{s-t}}\Big)\nu^{(16r_1+4)k}x^{(16r_1+4)p^{s-t}}+\frac{\sqrt{2}}{2}\left(1-\left(\sqrt{2}\left(\sqrt{2-\sqrt{2}}-1\right)+1\right)x^{8p^{s-t}}\right)\nu^{(16r_1+6)k}x^{(16r_1+6)p^{s-t}}+$
$c\frac{\sqrt{2}}{2}\sqrt{-\left(2-\sqrt{2-\sqrt{2}}\right)}\,(1+(\sqrt{2}-1)x^{8p^{s-t}})\nu^{(16r_1+7)k}x^{(16r_1+7)p^{s-t}}+$
$d\sqrt{2-\sqrt{2}}\sqrt{-\left(2-\sqrt{2-\sqrt{2}}\right)}\,\nu^{(16r_1+5)k}x^{(16r_1+13)p^{s-t}})x^{32r_2p^{l+s-t}}\Bigg\}$,

Where $a=-1, b=1, c=-1, d=-1$ for $m=24p^l$ and $a=1, b=-1, c=1, d=1$ for $26p^l$

10. $\phi_{m+k}^{(i)}(x)=\frac{1}{16p^{l+t}}\Bigg\{\sum_{r_1=0}^{2p^l-1}\sum_{r_2=0}^{p^t-1}(-1)^{r_1}\left(-\frac{\sqrt{2}}{2}\left(\sqrt{2-\sqrt{2}}-\sqrt{2}+1+\left(\sqrt{2-\sqrt{2}}+1\right)x^{8p^{s-t}}\right)\right)\nu^{16r_1i}x^{16r_1p^{s-t}}+$
$a\frac{\sqrt{2}\sqrt{2-\sqrt{2}}\sqrt{-(2+\sqrt{2-\sqrt{2}})}}{2}(1-x^{8p^{s-t}})\nu^{(16r_1+1)i}x^{(16r_1+1)p^{s-t}}+\frac{\sqrt{2}}{2}\Big(\sqrt{2-\sqrt{2}}-\sqrt{2}+1-\Big(\sqrt{2-\sqrt{2}}+$
$1\Big)x^{8p^{s-t}}\Big)\alpha^{(16r_1+2)i}x^{(16r_1+2)p^{s-t}}+b\frac{\sqrt{2}}{2}\sqrt{-(2+\sqrt{2-\sqrt{2}})}(\sqrt{2}-1+x^{8p^{s-t}})\nu^{(16r_1+3)i}x^{(16r_1+3)p^{s-t}}+\frac{\sqrt{2}}{2}\Big(\sqrt{2}\left(\sqrt{2-\sqrt{2}}+1\right)-$
$1+x^{8p^{s-t}}\Big)\nu^{(16r_1+4)k}x^{(16r_1+4)p^{s-t}}+\frac{\sqrt{2}}{2}\left(1+\left(\sqrt{2}\left(\sqrt{2-\sqrt{2}}+1\right)-1\right)x^{8p^{s-t}}\right)\nu^{(16r_1+6)k}x^{(16r_1+6)p^{s-t}}+$
$c\frac{\sqrt{2}}{2}\sqrt{-\left(2+\sqrt{2-\sqrt{2}}\right)}\,(1+(\sqrt{2}-$
$1)x^{8p^{s-t}})\nu^{(16r_1+7)k}x^{(16r_1+7)p^{s-t}}d\sqrt{2-\sqrt{2}}\sqrt{-\left(2+\sqrt{2-\sqrt{2}}\right)}\,\nu^{(16r_1+5)k}x^{(16r_1+13)p^{s-t}})x^{32r_2p^{l+s-t}}\Bigg\}$;where $a=1, b=1, c=-1, d=1$ for $m=28p^l$ and $a=-1, b=-1, c=1, d=-1$ for $30p^l$

and $\nu^i=\xi_{p^t}^i\alpha^c$

(3) For $t\geq s$, by lemma 4.2 (iii) [5], $x-\xi_{p^s}^{-i}\alpha^{-cp^{t-s}}$, $x+\xi_{p^s}^{-i}\alpha^{-cp^{t-s}}$, $x^2+\xi_{p^s}^{-i}\alpha^{-2cp^{t-s}}$, … , $x^2+\sqrt{-(2+\sqrt{2-\sqrt{2}})}\,\xi_{p^s}^{-i}\alpha^{-cp^{t-s}}x-\xi_{p^s}^{-2i}\alpha^{-2cp^{t-s}}$ are irreducible over $\mathbb{F}_q$ and the primitive idempotents in $\mathbb{F}_q[x]/\langle x^{16p^{l+s}}-1\rangle$ corresponding to these irreducible polynomials are respectively as follows :

$$\phi_k^{(i)}(x)=\frac{1}{32p^{l+s}}\left\{\sum_{r_1=0}^{p^s-1}\sum_{r_2=0}^{4p^l-1}\xi_{p^s}^{r_1i}\alpha^{-cr_1p^{t-s}+kr_2}x^{r_1+r_2p^s}\right\};m=0$$

$$\phi_{m+k}^{(i)}(x)=\frac{1}{32p^{l+s}}\left\{\sum_{r_1=0}^{p^s-1}\sum_{r_2=0}^{4p^l-1}(-1)^{r_1+r_2}\xi_{p^s}^{r_1i}\alpha^{-cr_1p^{t-s}+kr_2}x^{r_1+r_2p^s}\right\};m=p^s$$

$$\phi_{m+k}^{(i)}(x)=\frac{1}{16p^{l+s}}\left\{\sum_{r_1=0}^{p^s-1}\sum_{r_2=0}^{4p^l-1}(-1)^{r_1+r_2}\xi_{p^s}^{r_1i}\alpha^{-2cr_1p^{t-s}+2kr_2}x^{r_1+r_2p^s}\right\};\ m=2p^s$$

$$\phi_{m+k}^{(i)}(x)=\frac{1}{16p^{l+s}}\left\{\sum_{r_1=0}^{2p^{l+s-t}-1}\sum_{r_2=0}^{p^t-1}(-1)^{r_1}\left(\mu^{4r_1i}x^{4r_1}+\frac{\sqrt{-2}}{2}\mu^{(4r_1+3)i}x^{(4r_1+1)}(1-x^2)x^{32r_2p^{l+s-t}}\right)\right\};m=4p^s$$

$$\phi_{m+k}^{(i)}(x)=\frac{1}{16p^{l+s}}\left\{\sum_{r_1=0}^{2p^{l+s-t}}\sum_{r_2=0}^{p^t-1}(-1)^{r_1}\left(\mu^{4r_1i}x^{4r_1}-\frac{\sqrt{-2}}{2}\mu^{(4r_1+3)i}x^{(4r_1+1)}(1-x^2)x^{32r_2p^{l+s-t}}\right)\right\};m=6p^s$$

$$\phi_{m+k}^{(i)}(x)=\frac{1}{16p^{l+s}}\left\{\sum_{r_1=0}^{2p^{l+s-t}-1}\sum_{r_2=0}^{p^t-1}(-1)^{r_1}\left(\frac{\sqrt{2}}{2}(\sqrt{2}-1-x^{4p^{s-t}})\mu^{8r_1i}x^{8r_1}+a\frac{\sqrt{2}}{2}\sqrt{-(2-\sqrt{2})}\mu^{(8r_1+5)i}(1-x^4)x^{(8r_1+1)}\right.\right.$$
$$\left.\left.-\frac{\sqrt{2}}{2}(\sqrt{2}-1+x^4)\mu^{(8r_1+2)i}x^{(8r_1+2)}+b\sqrt{-(2-\sqrt{2})}\,\mu^{(8r_1+3)i}x^{(8r_1+7)}\right)x^{32r_2p^{l+s-t}}\right\};$$

where $a=1,\ b=-1$ for $m=8p^l$ and $a=-1,\ b=1$ for $m=10p^l$

$\phi_{m+k}^{(i)}(x)=\frac{1}{16p^{l+s}}\left\{\sum_{r_1=0}^{2p^{l+s-t}-1}\sum_{r_2=0}^{p^t-1}(-1)^{r_1}\left(\frac{\sqrt{2}}{2}(\sqrt{2}+1+x^{4p^{s-t}})\mu^{8r_1i}x^{8r_1}+a\frac{\sqrt{2}}{2}\sqrt{-(2+\sqrt{2})}\mu^{(8r_1+5)i}(1-x^4)x^{(8r_1+1)}-\right.\right.$ $\left.\left.\frac{\sqrt{2}}{2}(\sqrt{2}+1+x^4)\mu^{(8r_1+2)i}x^{(8r_1+2)}+b\sqrt{-(2+\sqrt{2})}\,\mu^{(8r_1+3)i}x^{(8r_1+7)}\right)x^{32r_2p^{l+s-t}}\right\}$; where $a=-1,\ b=-1$ for $m=12p^l$ and $a=1,\ b=1$ for $m=14p^l$

$\phi_{m+k}^{(i)}(x)=\frac{1}{16p^{l+s}}\left\{\sum_{r_1=0}^{2p^{l+s-t}-1}\sum_{r_2=0}^{p^t-1}(-1)^{r_1}\left(-\frac{\sqrt{2}}{2}\left(\sqrt{2+\sqrt{2}}-1-\sqrt{2}+\left(\sqrt{2+\sqrt{2}}-1\right)x^8\right)\right)\mu^{16r_1i}x^{16r_1}+\right.$ $a\frac{\sqrt{2}\sqrt{2+\sqrt{2}}\sqrt{-(2-\sqrt{2+\sqrt{2}})}}{2}(1-x^8)\mu^{(16r_1+1)i}x^{(16r_1+1)}+\frac{\sqrt{2}}{2}\left(\sqrt{2+\sqrt{2}}-1-\sqrt{2}-\left(\sqrt{2+\sqrt{2}}-1\right)x^8\right)\mu^{(16r_1+2)i}x^{(16r_1+2)}+$ $b\frac{\sqrt{2}}{2}\sqrt{-(2-\sqrt{2+\sqrt{2}})}(\sqrt{2}+1-x^8)\mu^{(16r_1+3)i}x^{(16r_1+3)}-\frac{\sqrt{2}}{2}\left(\sqrt{2}\left(\sqrt{2+\sqrt{2}}-1\right)-1+x^8\right)\mu^{(16r_1+4)k}x^{(16r_1+4)}-$ $\frac{\sqrt{2}}{2}\left(1+\left(\sqrt{2}\left(\sqrt{2+\sqrt{2}}-1\right)-1\right)x^8\right)\mu^{(16r_1+6)k}x^{(16r_1+6)}+c\frac{\sqrt{2}}{2}\sqrt{-\left(2-\sqrt{2+\sqrt{2}}\right)}\left(1-(\sqrt{2}+1)x^8\right)\mu^{(16r_1+7)k}x^{(16r_1+7)}+$ $\left.d\sqrt{2+\sqrt{2}}\sqrt{-\left(2-\sqrt{2+\sqrt{2}}\right)}\mu^{(16r_1+5)k}x^{(16r_1+13)})x^{32r_2p^{l+s-t}}\right\}$; where $a=1,b=1,c=1,d=-1$ for $m=16p^l$ and $a=-1,b=-1,c=-1,d=1$ for $18p^l$

$\phi_{m+k}^{(i)}(x)=\frac{1}{16p^{l+s}}\left\{\sum_{r_1=0}^{2p^{l+s-t}-1}\sum_{r_2=0}^{p^t-1}(-1)^{r_1}\left(\frac{\sqrt{2}}{2}\left(\sqrt{2+\sqrt{2}}+\sqrt{2}+1-\left(\sqrt{2+\sqrt{2}}+1\right)x^8\right)\right)\mu^{16r_1i}x^{16r_1}+\right.$ $a\frac{\sqrt{2}\sqrt{2+\sqrt{2}}\sqrt{-(2+\sqrt{2+\sqrt{2}})}}{2}(1-x^8)\mu^{(16r_1+1)i}x^{(16r_1+1)}-\frac{\sqrt{2}}{2}\left(\sqrt{2+\sqrt{2}}+\sqrt{2}+1-\left(\sqrt{2+\sqrt{2}}+1\right)x^8\right)\mu^{(16r_1+2)i}x^{(16r_1+2)}+$ $b\frac{\sqrt{2}}{2}\sqrt{-(2+\sqrt{2+\sqrt{2}})}(\sqrt{2}+1-x^8)\mu^{(16r_1+3)i}x^{(16r_1+3)}+\frac{\sqrt{2}}{2}\left(\sqrt{2}\left(\sqrt{2+\sqrt{2}}+1\right)+1-x^8\right)\mu^{(16r_1+4)k}x^{(16r_1+4)}-$ $\frac{\sqrt{2}}{2}\left(1-\left(\sqrt{2}\left(\sqrt{2+\sqrt{2}}+1\right)+1\right)x^8\right)\mu^{(16r_1+6)k}x^{(16r_1+6)}+c\frac{\sqrt{2}}{2}\sqrt{-\left(2+\sqrt{2+\sqrt{2}}\right)}\left(1-(\sqrt{2}+1)x^8\right)\mu^{(16r_1+7)k}x^{(16r_1+7)}+$ $\left.d\sqrt{2+\sqrt{2}}\sqrt{-\left(2+\sqrt{2+\sqrt{2}}\right)}\mu^{(16r_1+5)k}x^{(16r_1+13)})x^{32r_2p^{l+s-t}}\right\}$;

Where $a=-1,b=1,c=1,d=1$ for $m=20p^l$ and $a=1,b=-1,c=-1,d=-1$ for $22p^l$

$\phi_{m+k}^{(i)}(x)=\frac{1}{16p^{l+t}}\left\{\sum_{r_1=0}^{2p^{l+s-t}-1}\sum_{r_2=0}^{p^t-1}(-1)^{r_1}\left(\frac{\sqrt{2}}{2}\left(\sqrt{2-\sqrt{2}}+\sqrt{2}-1+\left(\sqrt{2-\sqrt{2}}-1\right)x^8\right)\right)\mu^{16r_1i}x^{16r_1}+\right.$ $a\frac{\sqrt{2}\sqrt{2-\sqrt{2}}\sqrt{-(2-\sqrt{2-\sqrt{2}})}}{2}(1-x^8)\mu^{(16r_1+1)i}x^{(16r_1+1)}-\frac{\sqrt{2}}{2}\left(\sqrt{2-\sqrt{2}}+\sqrt{2}-1-\left(\sqrt{2-\sqrt{2}}-1\right)x^8\right)\mu^{(16r_1+2)i}x^{(16r_1+2)}+$ $b\frac{\sqrt{2}}{2}\sqrt{-(2-\sqrt{2-\sqrt{2}})}(\sqrt{2}-1+x^8)\mu^{(16r_1+3)i}x^{(16r_1+3)}-\frac{\sqrt{2}}{2}\left(\sqrt{2}\left(\sqrt{2-\sqrt{2}}-1\right)+1+x^8\right)\mu^{(16r_1+4)k}x^{(16r_1+4)}+$ $\frac{\sqrt{2}}{2}\left(1-\left(\sqrt{2}\left(\sqrt{2-\sqrt{2}}-1\right)+1\right)x^8\right)\mu^{(16r_1+6)k}x^{(16r_1+6)}+c\frac{\sqrt{2}}{2}\sqrt{-\left(2-\sqrt{2-\sqrt{2}}\right)}\left(1+(\sqrt{2}-1)x^8\right)\mu^{(16r_1+7)k}x^{(16r_1+7)}+$ $\left.d\sqrt{2-\sqrt{2}}\sqrt{-\left(2-\sqrt{2-\sqrt{2}}\right)}\mu^{(16r_1+5)k}x^{(16r_1+13)})x^{32r_2p^{l+s-t}}\right\}$;

Where $a=-1,b=1,c=-1,d=-1$ for $m=24p^l$ and $a=1,b=-1,c=1,d=1$ for $26p^l$

$$\phi_{m+k}^{(i)}(x) = \frac{1}{16p^{l+t}}\left\{\sum_{r_1=0}^{2p^{l+s-t}-1}\sum_{r_2=0}^{p^t-1}(-1)^{r_1}\left(-\frac{\sqrt{2}}{2}\left(\sqrt{2-\sqrt{2}}-\sqrt{2}+1+\left(\sqrt{2-\sqrt{2}}+1\right)x^8\right)\right)\mu^{16r_1 i}x^{16r_1} + \frac{\sqrt{2}\sqrt{2-\sqrt{2}}\sqrt{-(2+\sqrt{2-\sqrt{2}})}}{2}(1-x^8)\mu^{(16r_1+1)i}x^{(16r_1+1)} + \frac{\sqrt{2}}{2}\left(\sqrt{2-\sqrt{2}}-\sqrt{2}+1-\left(\sqrt{2-\sqrt{2}}+1\right)x^8\right)\mu^{(16r_1+2)i}x^{(16r_1+2)} + \frac{\sqrt{2}}{2}\sqrt{-(2+\sqrt{2-\sqrt{2}})}(\sqrt{2}-1+x^8)\mu^{(16r_1+3)i}x^{(16r_1+3)} + \frac{\sqrt{2}}{2}\left(\sqrt{2}\left(\sqrt{2-\sqrt{2}}+1\right)-1+x^8\right)\mu^{(16r_1+4)k}x^{(16r_1+4)} + \frac{\sqrt{2}}{2}\left(1+\left(\sqrt{2}\left(\sqrt{2-\sqrt{2}}+1\right)-1\right)x^8\right)\mu^{(16r_1+6)k}x^{(16r_1+6)} - \frac{\sqrt{2}}{2}\sqrt{-\left(2+\sqrt{2-\sqrt{2}}\right)}\left(1+(\sqrt{2}-1)x^8\right)\mu^{(16r_1+7)k}x^{(16r_1+7)} + \sqrt{2-\sqrt{2}}\sqrt{-\left(2+\sqrt{2-\sqrt{2}}\right)}\mu^{(16r_1+5)k}x^{(16r_1+13)})x^{32r_2p^{l+s-t}}\right\};$$

Where $a = 1, b = 1, c = -1, d = 1$ for $m = 28p^l$ and $a = -1, b = -1, c = 1, d = -1$ for $30p^l$

and where $\mu^i = \xi_{p^s}^i \alpha^{cp^{t-s}}$

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