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p(x) - Gaussian Jacobsthal and p(x) - Gaussian Jacobsthal Lucas Polynomials

[a, #] **Arun Kumar Yadav**

[a] Department of Mathematics, IGU, Meerpur, Rewari, India

([#] Corresponding Author)

[b] **Manju Pruthi**

[b] Department of Mathematics, IGU, Meerpur, Rewari, India

Abstract: The purpose of this paper is to introduce the notions of p(x) - Gaussian Jacobsthal and p(x) - Gaussian Jacobsthal Lucas polynomials and to formulate some interesting results including the Generating function, Q-matrix and determinantal representations for these polynomials. Further, generalizations of Binet formula & Cassini's Identity for these notions are also obtained. This paper generalizes the work of Asci and Gurel [3] and extends the work of Catarino and Morgado [4].

Keywords: p(x) - Gaussian Jacobsthal polynomials; p(x) - Gaussian Jacobsthal Lucas Polynomials; Binet formula, Cassini's Identity.

1. Introduction

Berzsenyi [5] in 1977 propounded the theory of Gaussian Fibonacci number. Afterwards, various researchers worked on this theory and gave new innovations in it. Consequently, various interesting results related to Gaussian Fibonacci and Gaussian Lucas polynomials were formulated. Asci and Gurel [1] gave the notions of Bivariate Gaussian Fibonacci and Bivariate Gaussian Lucas Polynomials and thereby discussed generating functions, Binet formulas, explicit formulas and Q matrix for these polynomials. On the other hand, Asci and Gurel [2] defined Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers and formulated some interesting results using these notions. Subsequently, Asci and Gurel [3] gave Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomials and thereby discussed generating function, Binet formula, Explicit formulas and determinantal representations for these polynomials. Recently, Catarino and Morgado [4] generalized Jacobsthal and Jacobsthal Lucas polynomials by introducing the notions of h(x) – Jacobsthal and h(x) – Jacobsthal Lucas polynomials. In the same work [4], the authors have also discussed the properties of these polynomials and thereby provided generalization of the usual identities.

In the present paper inspired by the work of Catarino and Morgado [4], we formulate p(x) – Gaussian Jacobsthal and p(x) – Gaussian Jacobsthal Lucas polynomials & prove certain results involving these polynomials.

Next, we shall give some basic notions useful in our study.

In 1996, Horadam [10] innovated Jacobsthal and the Jacobsthal Lucas sequences using the following recurrence formulae:

$$\mathrm{J_n} = \mathrm{J_{n-1}} + 2\mathrm{J_{n-2}}, \ n \geq 2 \qquad (1.1)$$

where $J_0 = 0$ and $J_1 = 1$, and

$$\mathrm{j_n} = \mathrm{j_{n-1}} + 2\mathrm{j_{n-2}}, \ n \geq 2 \qquad (1.2)$$

where $j_0 = 2$ and $j_1 = 1$.

Horadam [11] also defined the notions of Jacobsthal and the Jacobsthal Lucas polynomial sequences as follows:

$$\mathrm{J_n}(x) = \mathrm{J_{n-1}}(x) + 2x\mathrm{J_{n-2}}(x), \ n \geq 2, \qquad (1.3)$$

where $J_0(x) = 0$ and $J_1(x) = 1$,

and
$$\mathrm{j_n}(x) = \mathrm{j_{n-1}}(x) + 2x\mathrm{j_{n-2}}(x), \ n \geq 2, \qquad (1.4)$$

where $j_0(x) = 2$ and $j_1(x) = 1$.

Afterwards, Asci and Gurel [2] defined the Gaussian Jacobsthal sequence and Gaussian Jacobsthal Lucas sequences as follows:

$$\mathrm{GJ_{n+1}} = G\mathrm{J_n} + 2\mathrm{GJ_{n-1}}, \ n \geq 1, \qquad (1.5)$$

where $GJ_0 = \frac{i}{2}$ and $GJ_1 = 1$,

and
$$\mathrm{Gj_{n+1}} = G\mathrm{j_n} + 2\mathrm{Gj_{n-1}}, \ n \geq 1, \qquad (1.6)$$

where $\mathrm{Gj_0} = 2 - \frac{i}{2}$ and $\mathrm{Gj_1} = 1 + 2i$.

It can be easily seen that $GJ_n = J_n + iJ_{n-1}$ and $Gj_n = j_n + ij_{n-1}$.

Later, Asci and Gurel [3] formulated Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomials by using the following recursion:

$$GJ_{n+1}(x) = GJ_n(x) + 2xGJ_{n-1}(x),\ n \geq 1, \quad (1.7)$$

where $GJ_0(x) = \frac{i}{2}$, $GJ_1(x) = 1$,

and
$$Gj_{n+1}(x) = Gj_n(x) + 2xGj_{n-1}(x),\ n \geq 1, \quad (1.8)$$

where $Gj_0(x) = 2 - \frac{i}{2}$ and $Gj_1(x) = 1+2ix$.

Again, it can also be seen that $GJ_n(x) = J_n(x) + ixJ_{n-1}(x)$ and $Gj_n(x) = j_n(x) + ixj_{n-1}(x)$.

On the other hand, Catarino and Morgado [4] gave the following notions of h(x) – Jacobsthal and h(x) – Jacobsthal Lucas polynomials:

$$J_{h,n+1}(x) = J_{h,n}(x) + h(x)J_{h,n-1}(x),\ n \geq 1, \quad (1.9)$$

where $J_{h,0}(x) = 0$ and $J_{h,1}(x) = 1$,

and
$$j_{h,n+1}(x) = j_{h,n}(x) + h(x)j_{h,n-1}(x),\ n \geq 1, \quad (1.10)$$

where $j_{h,0}(x) = 2$ and $j_{h,1}(x) = 1$.

Now we define our notions as follows:

Let us assume p(x) be a polynomial with real coefficients and $n \geq 1$ any integer.

Definition 1.1. The p(x) - Gaussian Jacobsthal polynomial sequence $\{GJ_{p,n}(x)\}_{n=0}^{\infty}$ is defined by the following recurrence relation:

$$GJ_{p,n+1}(x) = GJ_{p,n}(x) + p(x)\, GJ_{p,n-1}(x), \quad (1.11)$$

where $n \geq 1$, p is non – negative integer and the sequence $\{GJ_{p,n}(x)\}_{n=0}^{\infty}$ is along with the following initial conditions:

$GJ_{p,0}(x) = \frac{i}{2}$ and $GJ_{p,1}(x) = 1$.

Definition 1.2. The p(x) - Gaussian Jacobsthal Lucas polynomial sequence $\{Gj_{p,n}(x)\}_{n=0}^{\infty}$ is defined by the following recurrence relation:

$$Gj_{p,n+1}(x) = Gj_{p,n}(x) + p(x).\, Gj_{p,n-1}(x), \quad (1.12)$$

where $n \geq 1$, p is non – negative integer and the sequence $\{Gj_{p,n}(x)\}_{n=0}^{\infty}$ is along with the following initial conditions:

$Gj_{p,0}(x) = 2 - \frac{i}{2}$ and $Gj_{p,1}(x) = 1 + i\,p(x)$.

Next, we give few numbers of terms of these sequences in the following tables:

p(x) - Gaussian Jacobsthal Polynomials:

n	$\mathbf{GJ_{p,n}}$
0	$\frac{i}{2}$
1	1
2	$1+\frac{i}{2}p(x)$
3	$1+p(x)+\frac{i}{2}p(x)$
4	$1+2p(x)+\frac{i}{2}[p(x)+p^2(x)]$
5	$1+3p(x)+p^2(x)+\frac{i}{2}[p(x)+2p^2(x)]$
6	$1+4p(x)+3p^2(x)+\frac{i}{2}[p(x)+3p^2(x)+p^3(x)]$
⋮	⋮

p(x) - Gaussian Jacobsthal Lucas Polynomials:

n	$Gj_{p,n}$
0	$2 - \frac{i}{2}$
1	$1+ip(x)$
2	$1+ 2p(x) + \frac{i}{2}p(x)$
3	$1+3p(x) + \frac{i}{2}[p(x) + 2p^2(x)]$
4	$1+4p(x) + 2p^2(x) + \frac{i}{2}[p(x) + 3p^2(x)]$
5	$1 + 5p(x) + 5p^2(x) + \frac{i}{2}[p(x) + 4p^2(x) + 2p^3(x)]$
6	$1 + 6p(x) + 9p^2(x) + 2p^3(x) + \frac{i}{2}[p(x) + 5p^2(x) + 5p^3(x)]$
⋮	⋮

It is interesting to note that on considering p(x) = 2x in (1.11) and (1.12), (1.11) reduces to (1.7) and (1.12) reduces to (1.8), respectively.

In the next section, first we give some recursive properties of p(x) - Gaussian Jacobsthal and p(x) – Gaussian Jacobsthal Lucas polynomials and then obtain some important results.

Before we proceed, we shall consider the following characteristic equation:

$$t^2 - t - p(x) = 0. \quad (1.13)$$

The characteristic roots of (1.13) are defined by

$$t = \frac{1 \pm \sqrt{1+4p(x)}}{2}.$$

Let $\alpha(x) = \frac{1+\sqrt{1+4p(x)}}{2}$ and $\beta(x) = \frac{1-\sqrt{1+4p(x)}}{2}$,

then, $\alpha(x) + \beta(x) = 1$, $\alpha(x)\beta(x) = -p(x)$ and $\alpha(x) - \beta(x) = \sqrt{1+4p(x)}$.

2. Mains Results

In this section, we shall give our main results as follows:

Some Properties

Theorem 2.1. The Generating function for

(i) p(x) - Gaussian Jacobsthal polynomial is given as follows:

$$g(t,x) = \sum_{n=0}^{\infty} GJ_{p,n}(x)\, t^n = \frac{2t + i(1-t)}{2(1 - t - p(x)\, t^2)}$$

and

(ii) p(x) - Gaussian Jacobthal Lucas polynomial is given as follows:

$$h(t,x) = \sum_{n=0}^{\infty} Gj_{p,n}(x)\, t^n = \frac{4-2t+ i(t-1+2\,t\,p(x))}{2(1-t- p(x)\,t^2)}.$$

Proof. (i) Let g(t,x) be the generating function of p(x) - Gaussian Jacobsthal polynomial sequence $GJ_{p,n}(x)$, then

$$g(t,x) - t\,g(t,x) - t^2 p(x)\,g(t,x) = \sum_{n=0}^{\infty} GJ_{p,n}(x)\, t^n - \sum_{n=0}^{\infty} GJ_{p,n}(x)\, t^{n+1} - p(x)\sum_{n=0}^{\infty} GJ_{p,n}(x)\, t^{n+2}$$

$$= GJ_{p,0}(x) + GJ_{p,1}(x).t - GJ_{p,0}(x)\, t + \sum_{n=2}^{\infty} t^n[GJ_{p,n}(x) - GJ_{p,n-1}(x) - p(x)GJ_{p,n-2}(x)]$$

$$= \frac{i}{2} + t - \frac{i}{2}\, t$$

$$= t + \frac{i}{2}(1- t)$$

$= \frac{2t + i(1-t)}{2}$,

which implies that g(t,x) $= \frac{2t + i(1-t)}{2(1-t-p(x)t^2)}$.

(ii) Let h(t,x) be the generalized function of p(x) - Gaussian Jacobsthal Lucas polynomial sequence $Gj_{p,n}$ (x) then

h(t,x) − t h(t,x) − p(x) t^2 h(t,x) = $\sum_{n=0}^{\infty} Gj_{p,n}(x)\, t^n - \sum_{n=0}^{\infty} Gj_{p,n}(x)\, t^{n+1}$ − $p(x) \sum_{n=0}^{\infty} Gj_{p,n}(x)\, t^{n+2}$

$$= Gj_{p,0}(x) + Gj_{p,1}(x)\, t - Gj_{p,0}(x)\, t + \sum_{n=2}^{\infty} t^n [Gj_{p,n}(x) - Gj_{p,n-1}(x) - p(x)\, Gj_{p,n-2}(x)]$$

$$= 2 - \frac{i}{2} + (1 + i\,p(x))\, t - (2 - \frac{i}{2})t$$

$$= 2 - \frac{i}{2} + t + i\,p(x)\,t - 2t + \frac{i}{2}t$$

$$= 2 - t + \frac{i}{2}[2t\,p(x) - 1 + t],$$

which implies, h(t,x) = $\frac{4 - 2t + i[2tp(x) - 1 + t]}{2(1 - t - p(x)t^2)}$.

The following interesting result is a generalization of Binet Formula & can be obtained easily by using mathematical induction:

Theorem 2.2. For $n \geq 0$,

(i) $GJ_{k,n}(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} + i\frac{p(x)}{2}\frac{\alpha^{n-1}(x) - \beta^{n-1}(x)}{\alpha(x) - \beta(x)}$;

(ii) $Gj_{k,n}(x) = \alpha^n(x) + \beta^n(x) + i\frac{p(x)}{2}(\alpha^{n-1}(x) + \beta^{n-1}(x))$.

Theorem 2.3. Let $M_n(x)$ denote the n× n tridiagonal matrix as

$$M_n(x) = \begin{bmatrix} 1 & i & 0 & \cdots & 0 \\ -\frac{p(x)}{2} & 1 & p(x) & \ddots & \vdots \\ 0 & -1 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & p(x) \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, n \geq 1$$

and let $M_0(x) = \frac{i}{2}$.

Then, det $M_n(x)$ = $GJ_{p,n}(x)$, n ≥ 1

Proof. We shall prove the result by using mathematical induction on n.

For n = 1 and n = 2, it is easy to obtain that

det $M_1(x)$ = 1 = $GJ_{p,1}(x)$

and det $M_2(x) = 1 + \frac{i}{2}p(x) = GJ_{p,2}(x)$.

Assume that the result is true for n − 1 and n − 2, that is

det $M_{n-1}(x) = GJ_{p,n-1}(x)$

and det $M_{n-2}(x) = GJ_{p,n-2}(x)$.

Now, we shall prove that the result for n

For, det $M_n(x)$ = det $M_{n-1}(x)$ + p(x)det $M_{n-2}(x)$

$= GJ_{p,n-1}(x) + p(x)GJ_{k,n-2}(x)$

$= GJ_{p,n}(x)$.

Theorem 2.4. Let $A_n(x)$ denote the n× n tridiagonal matrix as

$$A_n(x) = \begin{bmatrix} 2 - \frac{i}{2} & \frac{p(x)}{4} - 1 & 0 & \cdots & 0 \\ 1 & \frac{i.p(x)}{2} & p(x) & \ddots & \vdots \\ 0 & -1 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & p(x) \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, n \geq 1.$$

Then, det $A_n(x)$ = $GJ_{p,n-1}(x)$, n ≥ 0.

Proof. We shall prove the result by using mathematical induction on n.

For n = 1 and n = 2, it is easy to obtain that

$$\det A_1(x) = 2 - \frac{i}{2} = Gj_{p,0}(x)$$

and $$\det A_2(x) = 1 + p(x)\, i = Gj_{p,1}(x).$$

Assume that the result is true for n − 1 and n – 2, that is

$$\det A_{n-1}(x) = Gj_{p,n-2}(x)$$

and $$\det A_{n-2}(x) = Gj_{p,n-3}(x).$$

Now, we shall prove that the theorem for n.

For,
$$\det A_n(x) = \det A_{n-1}(x) + p(x) \det A_{n-2}(x)$$
$$= Gj_{p,n-1}(x) + p(x)\, Gj_{p,n-2}(x)$$
$$= Gj_{p,n}(x).$$

Now, we introduce the matrices Q(x), P and R that plays the role of the Q-Matrix of Fibonacci numbers. Let Q(x), P and R denotes the 2×2 matrices defined as

$$Q(x) = \begin{bmatrix} 1 & p(x) \\ 1 & 0 \end{bmatrix},\ P = \begin{bmatrix} 1 + i\frac{p(x)}{2} & 1 \\ 1 & \frac{i}{2} \end{bmatrix},\ R = \begin{bmatrix} 1 + 2p(x) + \frac{i}{2}p(x) & 1 + ip(x) \\ 1 + ip(x) & 2 - \frac{i}{2} \end{bmatrix}.$$

Theorem 2.5. Let n $\geq$ 1, Then

$$Q^n(x)\, P = \begin{bmatrix} GJ_{p,n+2}(x) & GJ_{p,n+1}(x) \\ GJ_{p,n+1}(x) & GJ_{p,n}(x) \end{bmatrix}$$

where $GJ_{p,n}(x)$ is the nth p(x) - Gaussian Jacobsthal Polynomial.

Proof. We shall prove the result by using mathematical induction on n.

For n = 1,

$$Q(x)\, P = \begin{bmatrix} 1 & p(x) \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 + \frac{i}{2}p(x) & 1 \\ 1 & \frac{i}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 + p(x) + \frac{i}{2}.p(x) & 1 + \frac{i}{2}.p(x) \\ 1 + \frac{i}{2}.p(x) & 1 \end{bmatrix}$$

$$= \begin{bmatrix} GJ_{p,3}(x) & GJ_{p,2}(x) \\ GJ_{p,2}(x) & GJ_{p,1}(x) \end{bmatrix}$$

Assume that the result holds for n = t, that is

$$[Q(x)]^{.t}.P = \begin{bmatrix} 1 & p(x) \\ 1 & 0 \end{bmatrix}^t \begin{bmatrix} 1 + \frac{i}{2}p(x) & 1 \\ 1 & \frac{i}{2} \end{bmatrix} = \begin{bmatrix} GJ_{p,t+2}(x) & GJ_{p,t+1}(x) \\ GJ_{p,t+1}(x) & GJ_{p,t}(x) \end{bmatrix}$$

Now, we shall prove the theorem for n = t+1, we have

$$[Q(x)]^{t+1}.p = \begin{bmatrix} 1 & p(x) \\ 1 & 0 \end{bmatrix}^{t+1} \begin{bmatrix} 1 + \frac{i}{2}p(x) & 1 \\ 1 & \frac{i}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & p(x) \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & p(x) \\ 1 & 0 \end{bmatrix}^{t} \begin{bmatrix} 1 + \frac{i}{2}p(x) & 1 \\ 1 & \frac{i}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & p(x) \\ 1 & 0 \end{bmatrix} \begin{bmatrix} GJ_{p,t+2}(x) & GJ_{p,t+1}(x) \\ GJ_{p,t+1}(x) & GJ_{p,t}(x) \end{bmatrix}$$

$$= \begin{bmatrix} GJ_{p,t+3}(x) & GJ_{p,t+2}(x) \\ GJ_{p,t+2}(x) & GJ_{p,t+1}(x) \end{bmatrix}$$

Theorem 2.6. Let n ≥ 1, Then

$$Q^n(x)\ R = \begin{bmatrix} Gj_{p,n+2}(x) & Gj_{p,n+1}(x) \\ Gj_{p,n+1}(x) & Gj_{p,n}(x) \end{bmatrix}$$

where $Gj_{p,n}(x)$ is the nth p(x) - Gaussian Jacobsthal Lucas Polynomial.

Proof. We shall prove the result by using mathematical induction on n.

For n = 1, we have

$$Q(x)\ R = \begin{bmatrix} 1 & p(x) \\ 1 & 0 \end{bmatrix} . \begin{bmatrix} 1 + 2p(x) + \frac{i}{2}p(x) & 1 + ip(x) \\ 1 + ip(x) & 2 - \frac{i}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 2p(x) + \frac{i}{2}p(x) + p(x) + ip^2(x) & 1 + ip(x) + 2p(x) - \frac{i}{2}p(x) \\ 1 + 2p(x) + \frac{i}{2}p(x) & 1 + ip(x) \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 3p(x) + \frac{i}{2}[p(x) + 2p^2(x)] & 1 + 2p(x) + \frac{i}{2}p(x) \\ 1 + 2p(x) + \frac{i}{2}p(x) & 1 + ip(x) \end{bmatrix}$$

$$= \begin{bmatrix} Gj_{p,3}(x) & Gj_{p,2}(x) \\ Gj_{p,2}(x) & Gj_{p,1}(x) \end{bmatrix}.$$

Assume that the theorem holds for n = t, that is

$$[Q(x)]^t\ R = \begin{bmatrix} 1 & p(x) \\ 1 & 0 \end{bmatrix}^t \begin{bmatrix} 1 + 2p(x) + \frac{i}{2}p(x) & 1 + ip(x) \\ 1 + ip(x) & 2 - \frac{i}{2} \end{bmatrix} = \begin{bmatrix} Gj_{p,t+2}(x) & Gj_{p,t+1}(x) \\ Gj_{p,t+1}(x) & Gj_{p,t}(x) \end{bmatrix}.$$

Now, we shall prove that the theorem for n = t + 1,

For,

$$[Q(x)]^{t+1}\ R = \begin{bmatrix} 1 & p(x) \\ 1 & 0 \end{bmatrix}^{t+1} \begin{bmatrix} 1 + 2p(x) + \frac{i}{2}p(x) & 1 + ip(x) \\ 1 + ip(x) & 2 - \frac{i}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & p(x) \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & p(x) \\ 1 & 0 \end{bmatrix}^{t} \begin{bmatrix} 1 + 2p(x) + \frac{i}{2}p(x) & 1 + ip(x) \\ 1 + ip(x) & 2 - \frac{i}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & p(x) \\ 1 & 0 \end{bmatrix} \begin{bmatrix} Gj_{p,t+2}(x) & Gj_{p,t+1}(x) \\ Gj_{p,t+1}(x) & Gj_{p,t}(x) \end{bmatrix}$$

$$= \begin{bmatrix} Gj_{p,t+3}(x) & Gj_{p,t+2}(x) \\ Gj_{p,t+2}(x) & Gj_{p,t+1}(x) \end{bmatrix}.$$

Next, we give Generalization of Cassini's Identity for p(x) – Gaussian Jacobsthal & p(x) – Gaussian Jacobsthal Lucas polynomials:

Theorem 2.7. For n ≥ 1,

$$GJ_{p,n-1}(x)\ GJ_{p,n+1}(x) - GJ_{p,n}^2(x) = \frac{(-1)^n}{2}(p(x))^{n-1}(\frac{p(x)}{2} + 2 - i).$$

Proof. We can prove the result by matrix method. For, we need to evaluate some determinnats as follows:

$$\det Q^{n-1}(x) = \begin{vmatrix} 1 & p(x) \\ 1 & 0 \end{vmatrix}^{n-1} = [-p(x)]^{n-1}.$$

$$\det P = \begin{vmatrix} 1 + \frac{i}{2}\ p(x) & 1 \\ 1 & \frac{i}{2} \end{vmatrix} = -\frac{i}{2} - \frac{p(x)}{4} - 1 = -\frac{1}{2}[\frac{p(x)}{2} + 2 - i].$$

Also, by Theorem 2.5, we can obtain

$$[Q(x)]^{n-1}\ P = \begin{bmatrix} GJ_{p,n+1}(x) & GJ_{p,n}(x) \\ GJ_{p,n}(x) & GJ_{p,n-1}(x) \end{bmatrix}$$

so that, we have $\det(Q^{n-1}(x)\ P) = GJ_{p,n-1}(x)\ GJ_{p,n+1}(x) - GJ_{p,n}^2(x)$,

or $$GJ_{p,n-1}(x).\,GJ_{p,n+1}(x) - GJ_{p,n}^2(x) = \det(Q^{n-1}(x)\ \mathrm{P})$$
$$= \det(Q^{n-1}(x))\det(\mathrm{P})$$
$$= [-p(x)]^{n-1}\left(-\tfrac{1}{2}[\tfrac{p(x)}{2} + 2 - i]\right)$$
$$= \tfrac{(-1)^n}{2}[(p(x)]^{n-1}[\tfrac{p(x)}{2} + 2 - i].$$

Theorem 2.8. For n ≥ 1,

$$Gj_{p,n-1}(x)\ Gj_{p,n+1}(x) - Gj_{p,n}^2(x) = (-1)^n p(x)^{n-1}\left\{\frac{17}{4}p(x) + 1 - \frac{i}{2}[1 + 4i.\,p(x)]\right\}.$$

Proof. We shall prove the result by using matrix method. For this, we require the following determinants:

$$\det Q^{n-1}(x) = \begin{vmatrix}1 & p(x)\\ 1 & 0\end{vmatrix}^{n-1} = [-p(x)]^{n-1},$$

$$\det R = \begin{vmatrix}1 + 2\mathrm{p}(x) + \frac{i}{2}p(x) & 1 + i\ \mathrm{p}(x)\\ 1 + i\,p(x) & 2 - \frac{i}{2}\end{vmatrix} = \frac{17}{4}\ p(x) + 1 - \frac{i}{2}[1 + 4i\ p(x)].$$

Also, by Theorem 2.6, we have

$$\mathrm{Q}^{n-1}(\mathrm{x})\ \mathrm{R} = \begin{bmatrix}\mathrm{Gj}_{p,n+1}(\mathrm{x}) & \mathrm{Gj}_{p,n}(\mathrm{x})\\ \mathrm{Gj}_{p,n}(\mathrm{x}) & \mathrm{Gj}_{p,n-1}(\mathrm{x})\end{bmatrix},$$

which implies $\det(Q^{n-1}(x)\ \mathrm{R}) = Gj_{p,n-1}(x)\ Gj_{p,n+1}(x) - Gj_{p,n}^2(x)$,

so that

$$Gj_{p,n-1}(x)\ Gj_{p,n+1}(x) - Gj_{p,n}^2(x) = \det(Q^{n-1}(x)\ \mathrm{R})$$
$$= \det(Q^{n-1}(x))\det(\mathrm{R})$$
$$= [-p(x)]^{n-1}\left\{\frac{17}{4}p(x) + 1 - \frac{i}{2}[1 + 4i\ p(x)]\right\}$$
$$= (-1)^{n-1}\ p(x)^{n-1}\left\{\frac{17}{4}p(x) + 1 - \frac{i}{2}[1 + 4i\ p(x)]\right\}.$$

Theorem 2.9. The sums of the p(x) - Gaussian Jacobsthal polynomial and p(x) - Gaussian Jacobsthal Lucas polynomial are given as:

(i) $\sum_{m=0}^{n} GJ_{p,m}(x) = \frac{1}{p(x)}\left[GJ_{p,n+2}(x) - 1\right];$

(ii) $\sum_{m=0}^{n} Gj_{p,m}(x) = \frac{1}{p(x)}\left[Gj_{p,n+2}(x) - (1 + i\ p(x))\right].$

Proof. (i) For n ≥ 1, we have

$$\mathrm{GJ}_{p,n+1}(\mathrm{x}) = \mathrm{GJ}_{p,n}(\mathrm{x}) + \mathrm{p(x)}\ \mathrm{GJ}_{p,n-1}(\mathrm{x}),$$
$$\mathrm{GJ}_{p,n-1}(\mathrm{x}) = \frac{1}{p(x)}\left[\mathrm{GJ}_{p,n+1}(\mathrm{x}) - \mathrm{GJ}_{p,n}(\mathrm{x})\right].$$

Taking n = 1, 2, 3, … , n + 1 in last equation, we can get

$$\mathrm{GJ}_{p,0}(\mathrm{x}) = \frac{1}{p(x)}\left[\mathrm{GJ}_{p,2}(\mathrm{x}) - \mathrm{GJ}_{p,1}(\mathrm{x})\right],$$
$$\mathrm{GJ}_{p,1}(\mathrm{x}) = \frac{1}{p(x)}\left[\mathrm{GJ}_{p,3}(\mathrm{x}) - \mathrm{GJ}_{p,2}(\mathrm{x})\right],$$
$$\mathrm{GJ}_{p,2}(\mathrm{x}) = \frac{1}{p(x)}\left[\mathrm{GJ}_{p,4}(\mathrm{x}) - \mathrm{GJ}_{p,3}(\mathrm{x})\right],$$
$$\vdots \qquad \vdots$$
$$\mathrm{GJ}_{p,n-1}(\mathrm{x}) = \frac{1}{p(x)}\left[\mathrm{GJ}_{p,n+1}(\mathrm{x}) - \mathrm{GJ}_{p,n}(\mathrm{x})\right],$$
$$\mathrm{GJ}_{p,n}(\mathrm{x}) = \frac{1}{p(x)}\left[\mathrm{GJ}_{p,n+2}(\mathrm{x}) - \mathrm{GJ}_{p,n+1}(\mathrm{x})\right].$$

On adding all of the above (n + 1) equations, we can get

$$\sum_{m=0}^{n} GJ_{p,m}(x) = \frac{1}{p(x)}\left[GJ_{p,n+2}(x) - GJ_{p,1}(x)\right]$$

or $$\sum_{m=0}^{n} GJ_{p,m}(x) = \frac{1}{\mathrm{p(x)}}\left[GJ_{k,n+2}(x) - 1\right].$$

(ii) For n ≥ 1, we have

$$\mathrm{Gj}_{\mathrm{p,n+1}}(\mathrm{x}) = \mathrm{Gj}_{\mathrm{p,n}}(\mathrm{x}) + \mathrm{p(x)}\,\mathrm{Gj}_{\mathrm{p,n-1}}(\mathrm{x});$$

$$\mathrm{Gj}_{\mathrm{p,n-1}}(\mathrm{x}) = \frac{1}{p(x)}\left[\mathrm{Gj}_{\mathrm{p,n+1}}(\mathrm{x}) - \mathrm{Gj}_{\mathrm{p,n}}(\mathrm{x})\right].$$

Taking n = 1, 2, 3, … , n + 1 in last equation, we can get

$$\mathrm{Gj}_{\mathrm{p,0}}(\mathrm{x}) = \frac{1}{p(x)}\left[\mathrm{Gj}_{\mathrm{p,2}}(\mathrm{x}) - \mathrm{Gj}_{\mathrm{p,1}}(\mathrm{x})\right],$$

$$\mathrm{Gj}_{\mathrm{p,1}}(\mathrm{x}) = \frac{1}{p(x)}\left[\mathrm{Gj}_{\mathrm{p,3}}(\mathrm{x}) - \mathrm{Gj}_{\mathrm{p,2}}(\mathrm{x})\right],$$

$$\mathrm{Gj}_{\mathrm{p,2}}(\mathrm{x}) = \frac{1}{p(x)}\left[\mathrm{Gj}_{\mathrm{p,4}}(\mathrm{x}) - \mathrm{Gj}_{\mathrm{p,3}}(\mathrm{x})\right],$$

$$\vdots \qquad\qquad \vdots$$

$$\mathrm{Gj}_{\mathrm{p,n-1}}(\mathrm{x}) = \frac{1}{p(x)}\left[\mathrm{Gj}_{\mathrm{p,n+1}}(\mathrm{x}) - \mathrm{Gj}_{\mathrm{p,n}}(\mathrm{x})\right],$$

$$\mathrm{Gj}_{\mathrm{p,n}}(\mathrm{x}) = \frac{1}{p(x)}\left[\mathrm{Gj}_{\mathrm{p,n+2}}(\mathrm{x}) - \mathrm{Gj}_{\mathrm{p,n+1}}(\mathrm{x})\right].$$

On adding all of the above (n + 1) equations, we can get

$$\sum_{m=0}^{n} Gj_{p,m}(x) = \frac{1}{p(x)}\left[Gj_{p,n+2}(x) - Gj_{p,1}(x)\right],$$

or $$\sum_{m=0}^{n} Gj_{p,m}(x) = \frac{1}{p(x)}\left[Gj_{p,n+2}(x) - (1 + i\,p(x))\right].$$

Conclusion:

The paper discusses new notions of p(x) – Gaussian Jacobsthal & p(x) – Gaussian Jacobsthal Lucas polynomials and provides Generating function, Q – matrix & determinantal representation for these polynomials, there by generalizing the work of Asci & Gurel [3]. Further, a generalized form of Binet formula & Cassini's Identity for these polynomials are also discussed. Present paper provides an extension of the work of Catarino & Morgado [4].

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