SOLVING SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS USING FOURIER TRANSFORMATION

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ABSTRACT:

The Fourier Transformation is beneficial in differential equations since it can formulate them as problems which are easier to solve. The Fourier Transform converts waveform data in the time domain into the frequency domain. It decomposes functions depending on space or time into functions depending on spatial or temporal frequency. In this paper we will discuss how to approach Fourier transform to Simultaneous Linear Differential Equations.

Keywords: Fourier Transform, Differential Equations.

INTRODUCTION

The Fourier Transform was invented by a French Mathematician Joseph Fourier in 1800. He has exerted strong influence on Mathematical physics through his theorie analytic de la chaleur. His techniques turned out to have many applications beside heat flow, in particular, forming the basis of modern music synthesizers and MP3players. The output of the transformation represents the image in the Fourier or frequency domain, while the input image is the spatial domain equivalent. It converts complex curves into sum of a series of cosine waves.

It is widely used in image analysis, image filtering, image reconstruction and image compression. The Fourier transform is an important image processing tool which is used to decompose an image into its sine and cosine components. In the field of communication a signal is generally a voltage and Fourier transform provides us an inside view of signal and its different domain, how it behaves when it passes through various communication channels, filters and amplifiers. It is used in designing electrical circuits and signal analysis.

PRELIMINARIES

DEFINITION:

Let f be a function. The Fourier Transform of f is defined by

$$F(f(s)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ist}dt$$

Where s is real[1].

DEFINITION:

The Inverse Fourier Transform $F(s)$ denoted by $F^{-1}F(s)$ is defined by

$$F^{-1}F(S) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(s)e^{-isx}dx$$ [3]

DEFINITION:

The Convolution of two functions $f(x)$ and $g(x)$ denoted by $f * g$ is defined

$$h(x) = f * g\ (x)$$
$$= \int_{-\infty}^{\infty} f(s)g(x-s)ds$$

$$= \int_{-\infty}^{\infty} f(x-s)g(s)ds$$ [5]

DEFINITION:

Fourier sine transform of the function $f(x)$ is defined by

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) sinsx\ dx = F_s(s)$$ [2]

DEFINITION:

The inverse Fourier sine transform of $F_s(s)$ is defined by

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F(s) sinsx\ ds$$ [2]

DEFINITION:

Fourier cosine transform of the function $f(x)$ is defined by

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) cossx\ dx = F_c\ (s)$$ [2]

DEFINITION:

The Inverse Fourier cosine transform of $F_c(s)$ is defined b

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c\ cossx\ ds$$ [2]

The wave form $w(t)$ is Fourier transformable if it satisfies both Dirichlet conditions:

i) Over any time interval of finite length, the function $w(t)$ is single valued with the finite number of maxima and minima and the number of discontinuous (if any) is finite.

ii) $W(t)$ is absolutely integrable.

i.e.) $\int_{-\infty}^{\infty} |w(t)| dt < \infty$

Where the above conditions are sufficient, but not necessary.

Property:

Fourier Transform is said to be Linear if

$$F[af(x) + bg(x)] = aF[f(x)] + bF[g(x)]$$

Proof:

We know that

$$[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\ dx$$

$$\therefore F[af(x) + bg(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |[af(x) + bg(x)] e^{isx}|\ ds$$

$$= a \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\ dx + b \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{isx}\ dx$$

$$= aF[f(x) + bF[g(x)]$$

Thus the Fourier Transform is Linear.

2. PROBLEM

1. $\frac{dy}{dx} - \frac{dy}{dt} = e^{kit}$ ……………….(1)

$\frac{dy}{dt} + x = 0$ $(-\infty < x < \infty), t > 0$ ..……………….(2)

Solution:

Take Fourier Transform on both side we obtain

$$F\left[\frac{dx}{dt}\right] - F\left[\frac{dy}{dt}\right] = F\left[e^{kit}\right] \qquad \text{....................(3)}$$

$$-i\alpha x(\alpha) + i\alpha y(\alpha) = F\left[e^{kit}\right]$$

$$F^{-1}\left[\sqrt{2\pi}\,\delta(\alpha)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{2\pi}\,\delta(\alpha) e^{i\alpha t} d\alpha$$

$$-i\alpha x(\alpha) + i\alpha y(\alpha) = \sqrt{2\pi}\delta(\alpha = k) \qquad \text{.....................(4)}$$

Taking Fourier Transform on (2)

$$F\left[\frac{dy}{dx}\right] + F[x] = 0$$

$$-i\alpha y(\alpha) + x(\alpha) = 0 \qquad \text{....................(5)}$$

From (4) and (5) we obtain

$$x(\alpha) = \frac{\sqrt{2\pi}\,\delta(\alpha = k)}{-i\alpha + 1}$$

$$F[y(t)] = x(\alpha) = \sqrt{2\pi}\frac{\delta(\alpha + k)}{-i\alpha + 1}$$

Apply Inverse Fourier Transform

$$x(t) = \sqrt{2\pi}\left\{F^{-1}\left[\frac{\delta(\alpha + k)}{-i\alpha + 1}\right]\right\}$$

Using Convolution Theorem

$$F^{-1}[F(\alpha).G(\alpha)] = f * g$$

$$F^{-1}\left[\frac{\delta(\alpha + k)}{-i\alpha + 1}\right] = F^{-1}\left[\delta(\alpha + k).\frac{1}{-i\alpha + 1}\right]$$

Solving the integral

$$F^{-1}\left[\frac{\delta(\alpha + k)}{-i\alpha + 1}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{iku}\left(\sqrt{2\pi}\right) e^{t-u} p(t-u)\ du$$

$$x(t) = \frac{e^{ikt}}{\sqrt{2\pi}\,(ik - 1)} \qquad \text{..................(6)}$$

From (4) and (5) we obtain

$$y(\alpha) = \frac{\sqrt{2\pi}\,\delta(\alpha + k)}{\alpha(\alpha + i)}$$

$$F[y(t)] = y(\alpha) = \frac{\sqrt{2\pi}}{i}\left[\frac{\delta(\alpha + k)}{\alpha} - \frac{\delta(\alpha + k)}{\alpha + i}\right]$$

Apply Inverse Fourier Transform

$$y(t) = \frac{\sqrt{2\pi}}{i}\left\{F^{-1}\left[\frac{\delta(\alpha + k)}{\alpha}\right] - F^{-1}\left[\frac{\delta(\alpha + k)}{\alpha + i}\right]\right\} \qquad \text{....................(7)}$$

Using Convolution Theorem

$$F^{-1}\left[\frac{\delta(\alpha + k)}{\alpha}\right] - F^{-1}\left[\frac{\delta(\alpha + k)}{\alpha + i}\right] = F^{-1}\left[\delta(\alpha + k).\frac{1}{\alpha}\right] - F^{-1}\left[\delta(\alpha + k).\frac{1}{\alpha + i}\right]$$

Solving the integral

$$F^{-1}\left[\frac{\delta(\alpha + k)}{\alpha}\right] - F^{-1}\left[\frac{\delta(\alpha + k)}{\alpha + i}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iku}\ du - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(ik-1)u}\ du$$

$$F^{-1}\left[\frac{\delta(\alpha + k)}{\alpha}\right] - F^{-1}\left[\frac{\delta(\alpha + k)}{\alpha + i}\right] = -\frac{1}{\sqrt{2\pi}}\left(\frac{e^{ikt}}{ik}\right) - \frac{1}{\sqrt{2\pi}}\left(\frac{e^{(ik-1)t}}{ik - 1}\right)$$

$$y(t) = \frac{\sqrt{2\pi}}{i}\left\{-\frac{1}{\sqrt{2\pi}}\left(\frac{e^{ikt}}{ik}\right) - \frac{1}{\sqrt{2\pi}}\left(\frac{e^{(ik-1)t}}{ik-1}\right)\right\}$$

$$y(t) = -\frac{1}{i}\left\{\frac{e^{ikt}}{ik} - \frac{e^{(ik-1)t}}{ik-1}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(8)$$

From (6) and (8) we obtain

$$x(t) = \frac{e^{ikt}}{\sqrt{2\pi}(ik-1)}$$

$$y(t) = -\frac{1}{i}\left\{\frac{e^{ikt}}{ik} - \frac{e^{(ik-1)t}}{ik-1}\right\}$$

2. $$\frac{dx}{dt} + y = sin2\pi\mu t \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\frac{dy}{dt} + x = \cos 2\pi\mu t \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Solution:

Taking Fourier Transform

$$F\left[\frac{dx}{dt}\right] + F[y] = F[sin2\pi\mu t]$$

$$-i\alpha x(\alpha) + y(\alpha) = \frac{1}{2i}[\delta(f-\mu) - \delta(f+\mu)] \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$F\left[\frac{dy}{dt}\right] + F[x] = F[\cos 2\pi\mu t]$$

$$-i\alpha y(\alpha) + x(\alpha) = \frac{1}{2}[\delta(f-\mu) + \delta(f+\mu)] \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

From (3) and (4) we obtain

$$x(\alpha)[\alpha^2+1] = \alpha\left[\frac{\delta(f-\mu) - \delta(f+\mu)}{2}\right] + \left[\frac{\delta(f-\mu) + \delta(f+\mu)}{2}\right]$$

Using Euler's formula

$$x(\alpha)[\alpha^2+1] = \alpha\left\{\frac{1}{2}e^{i2\pi\mu t} - \frac{1}{2}e^{-i2\pi\mu t}\right\} + \left\{\frac{1}{2}e^{i2\pi\mu t} + \frac{1}{2}e^{-i2\pi\mu t}\right\}$$

$$x(\alpha) = \frac{1}{2[\alpha^2+1]}\left\{e^{i2\pi\mu t}(\alpha+1) - e^{-i2\pi\mu t}(\alpha-1)\right\}$$

$$x(\alpha) = \frac{1}{\alpha^2+1}$$

$$x(t) = \frac{1}{2i}\left[\left(\frac{1}{\alpha-i}\right) - \left(\frac{1}{\alpha+i}\right)\right]$$

Apply inverse Fourier transform

$$x(t) = \frac{1}{2i}\left\{F^{-1}\left(\frac{1}{\alpha-i}\right) - F^{-1}\left(\frac{1}{\alpha+i}\right)\right\}$$

Using Convolution Theorem

$$F^{-1}\left[\frac{1}{\alpha+i}\right] = e^{-at}u(t)$$

Solving the integral

$$x(t) = \frac{1}{2i}\left\{\frac{1}{\sqrt{2\pi}}\int_t^{\infty}\left[\sqrt{2\pi}\, e^{-at}u(t) - \sqrt{2\pi}\, e^{at}u(t)\right]dt\right\}$$

$$x(t) = \frac{1}{ai}[\sin at] \qquad \ldots\ldots\ldots\ldots\ldots\ldots..(5)$$

From (3) and (4) we obtain

$$y(\alpha)[\alpha^2+1] = \frac{1}{2i}[\delta(f-\mu) - \delta(f+\mu)] + \frac{i\alpha}{2}[\delta(f-\mu) + \delta(f+\mu)]$$

Using Euler's formula

$$y(\alpha)[\alpha^2+1] = \frac{1}{i}\left[\frac{1}{2}e^{i2\pi\mu t} - \frac{1}{2}e^{i2\pi\mu t}\right] + i\alpha\left[\frac{1}{2}e^{i2\pi\mu t} + \frac{1}{2}e^{-i2\pi\mu t}\right]$$

$$y(\alpha) = \frac{i\alpha}{\alpha^2+1}$$

$$y(t) = \frac{i}{-2i}\left[\left(\frac{\alpha}{\alpha+i}\right) - \left(\frac{\alpha}{\alpha-i}\right)\right]$$

Apply Inverse Fourier Transform

$$y(t) = \frac{i}{-2i}F^{-1}\left\{\left(\frac{\alpha}{\alpha+i}\right) - \left(\frac{\alpha}{\alpha-i}\right)\right\}$$

Using Convolution Theorem

$$F^{-1}\left[\frac{\alpha}{\alpha+i}\right] - F^{-1}\left[\frac{\alpha}{\alpha-i}\right] = e^{-t} - e^{t}$$

Thus

$$y(t) = \frac{i}{-2i}[e^{-t} - e^{t}]$$

$$y(t) = \frac{e^{t} - e^{-t}}{2}$$

$$y(t) = \sin t \qquad \text{……………….(6)}$$

From (5) and (6) we obtain

$$x(t) = \frac{1}{ai}\sin at$$

$$y(t) = \sin t$$

Conclusion:

The main motive of this paper is to give a brief idea about implementation of Fourier Transformation in various fields and how to solve the simultaneous linear differential equations using Fourier Transform. It can be used in interpolate functions and to smooth signals.

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