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SOME DISCUSION ON PROPERTIES OF PARTIAL WRONSKIAN

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Abstract: In this paper, the authors discuss the fundamental solution of partial differential equation by wronskian method. We investigate the linear dependence and independence solution of a partial differential equation using partial Wronskian. We introduced some properties of partial wronskian and verified the results. Furthermore, we establish a theorem and investigate the result on a linear partial differential equations using partial Wronskian.

Keywords: Wronskian, Partial Wronskian, partial differential equation, determinant of a matrix. .

1. Introduction:

A differential equation is a type of equation which links between an unknown function and its derivatives. These equations are very important in many branches of science; mathematics, chemistry, physics, biochemistry, economics. Differential equation gives two types of solutions these are, the general solution and the particular solution of a differential equation. In the field of mathematical modeling of physical systems the rule of differential equations plays an important role. The main aim of the differential equation is for studying the nature of solutions that satisfy the differential equations. The solution of a partial differential equation is an expression for the dependent variable in terms of the independent variable which satisfies the differential equation. In this paper we discuss about partial wronskian to help us determine whether the set of solutions of a linear partial differential equations be a fundamental set of solutions. The term Wronskian is a practical way of determining whether a set of solution of a differential equation is independent or dependent solution.

2. Wronskian:

The word Wronskian was first introduced by Józef Hoene-Wroński (1776) and named by Thomas Muir (1882). Wronskian is used in the study of differential equations; it is a very important tool for checking whether the nature of solutions is a linear independent or dependent set of solutions. In [4] the author discussed the theorems and properties of partial Wronskian. In [6] we discussed theorems and properties of Wronskian. In this paper we extend our work on partial Wronskian.

Suppose that y_1(x) and y_2(x) are linearly independent solution of a differential equation, then the Wronskian is non vanishing,

$$W = \begin{vmatrix} y_1 & y_2 \\ {y_1}' & {y_2}' \end{vmatrix} = y_1\, {y_2}' - y_2 {y_1}' \neq 0$$

If y_1(x) and y_2(x) are linearly dependent solution of a differential equation, then

$$W\,[y_1,\ y_2] = 0$$

3. Partial Wronskian:

Partial Wronskian is used to investigate the dependence or independence solution of a partial differential equation. In this section, we discuss some properties and theorem of partial Wronskian.

Definition3.1: Let ϕ and **ψ** be any two solutions of a partial differential equation of variables x and y which are defined in the region R, then the partial Wronskian of ϕ and **ψ** is defined by, $\overline{W}\,[\phi, \boldsymbol{\psi}] = \begin{vmatrix} \phi & \psi \\ D(\phi) & D(\psi) \end{vmatrix}$

Where, $D(\phi) = (\frac{\partial}{\partial x} + \frac{\partial}{\partial y})\,(\phi(x, y))$ and $D(\psi) = (\frac{\partial}{\partial x} + \frac{\partial}{\partial y})\,(\boldsymbol{\psi}(x, y))$

If ϕ(x, y) and **ψ**(x, y) are linearly independent solution then $\overline{W}\,[\phi, \psi\ ] \neq 0$

Example:

Let $\phi = e^{ax}\sin bx$ and $\psi = e^{ax}\cos bx$ are two solution where a and b are constants and b ≠ 0

Then, $\overline{W}\,[\phi, \psi] = \overline{W}\,[e^{ax}\sin bx, e^{ax}\cos bx\,]$

$$= \begin{vmatrix} e^{ax}\sin bx & e^{ax}\cos bx \\ D(\,e^{ax}\sin bx\,) & D(\,e^{ax}\cos bx\,) \end{vmatrix} \quad (1)$$

Where, $D(e^{ax}\sin bx) = (\frac{\partial}{\partial x}+\frac{\partial}{\partial y}) (e^{ax}\sin bx)$

$= ae^{ax}\sin bx + be^{ax}\cos bx$, (2)

And $D(e^{ax}\cos bx) = (\frac{\partial}{\partial x}+\frac{\partial}{\partial y}) (e^{ax}\cos bx)$

$= ae^{ax}\cos bx - be^{ax}\sin bx$ (3)

From equation (1), (2) and (3) we have,

$\bar{W} [e^{ax}\sin bx, e^{ax}\cos bx] = - be^{2ax} \neq 0$

Hence the function $e^{ax}\sin bx$ and $e^{ax}\cos bx$ are the linearly independent functions.

Definition: 3.2. Let ϕ and **ψ** be any two functions of variables x and y which are defined in the region R, which are partial differentiable with respect to x and y on the region R are said to be linearly dependent on R then, $\bar{W}$ [ϕ, **ψ**] = 0

Example: Let ϕ = sinx + cosy and **ψ**= cosx + siny are two functions

Then, $\bar{W}$ [ϕ, **ψ**] = $\bar{W}$ [sinx + cosy, cosx + siny]

$$= \begin{vmatrix} \text{sinx} + \text{cosy} & \text{cosx} + \text{siny} \\ D(\text{sinx} + \text{cosy}) & D(\text{cosx} + \text{siny}) \end{vmatrix} \quad (4)$$

$D(\text{sinx} + \text{cosy}) = (\frac{\partial}{\partial x}+\frac{\partial}{\partial y}) (\text{sinx} + \text{cosy})$

= cosx - sinx (5)

And, $D(\text{cosx} + \text{siny}) = (\frac{\partial}{\partial x}+\frac{\partial}{\partial y}) (\text{cosx} + \text{siny})$

= - sinx + cosx (6)

Now from equation (4), (5) and (6) we have,

$\bar{W}$ [sinx + cosy , cosx + siny] = 0

Hence the function sinx + cosy and cosx + siny are the linearly dependent functions.

4. Fundamental set of solutions:

Let us consider partial differential equations,

$(D^2 - D'^2 + D - D') z = 0$ (7)

Now the general solution of (7) can be obtained by,

y (x) = c_1 f(x) + c_2 g(x), where c_1 and c_2 are constants. (8)

Now if, W [f, g] ≠ 0, then the solutions f(x) and g(x) are linearly independent solutions and form a fundamental set of solution and therefore the general solution (8) is a fundamental solution.

4.1. Examples on fundamental set of solution of partial differential equation using Wronskain method.

Example: Consider a linear homogeneous partial differential equations,

$D^4 - 2D^3D' + 2DD^3 - D'^4 = 0$ (9)

A. E, $m^4 - 2m^3 + 2m - 1 = 0$

=> $m^4 -1 - 2m^3 + 2m = 0$

=> $(m^2 -1) [m^2 + 1 -2m] = 0$

=> $(m + 1) (m - 1)^3 = 0$

=> m = -1, 1, 1, 1.

The solution of (9) can be written as

$$y = c_1(y - x) + c_2 (y + x) + c_3x(y + x) + c_4x^2(y + x)$$

Now, let, $\phi_1(x,y)$= y - x, $\phi_2(x,y)$ = y + x, $\phi_3(x,y)$= x(y + x), and $\phi_2(x,y) = x^2$ (y + x)

are the arbitrary function.

Now,

W $[\phi_1\phi_2\phi_3\phi_4]$

$$= \begin{vmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 \\ D(\phi_1) & D(\phi_2) & D(\phi_3) & D(\phi_4) \\ D^2(\phi_1) & D^2(\phi_2) & D^2(\phi_3) & D^2(\phi_4) \\ D^3(\phi_1) & D^3(\phi_2) & D^3(\phi_3) & D^3(\phi_4) \end{vmatrix}, \qquad D(\phi) = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)(\phi)$$

$$= \begin{vmatrix} y-x & y+x & x(y+x) & x^2(y+x) \\ 0 & 2 & y+3x & 2xy+4x^2 \\ 0 & 0 & 4 & 2y+10x \\ 0 & 0 & 0 & 12 \end{vmatrix}$$

$= 96\,(y - x) \neq 0$

Since, $W[\phi_1\phi_2\phi_3\phi_4] = 96\,(y - x) \neq 0$

Hence, $y = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + c_4\phi_4$, is the fundamental solution of the partial differential equation (9).

5. Some properties of Partial Wronskian:

5.1. $\overline{W}[\phi, \boldsymbol{\psi}] = -\overline{W}[\boldsymbol{\psi}, \phi]$

5.2. $\overline{W}[0, \boldsymbol{\psi}] = 0$

5.3. $\overline{W}[\phi, \phi] = 0$

5.4. $\overline{W}[1, \phi] = D(\phi)$ where $D(\phi) = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)(\phi)$

5.5. $\overline{W}[\phi + \alpha, \boldsymbol{\psi} + \alpha] = \overline{W}[\phi, \boldsymbol{\psi}] + \alpha\, D\, \overline{W}[\phi - \boldsymbol{\psi}]$

5.6. $\phi_1\, \overline{W}[\frac{\phi_2}{\phi_1}, \phi_1] + \phi_2\, \overline{W}[\frac{\phi_1}{\phi_2}, \phi_2] = D(\phi_1\phi_2)$

<u>**Verification of property**</u> <u>5.4</u>:

$\overline{W}[1, \phi] = D(\phi)$ where $D(\phi) = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)(\phi)$

Let, $\phi = e^{\alpha x}\cos\beta y$

Now, $\overline{W}[1, \phi] = \begin{vmatrix} 1 & e^{\alpha x}\cos\beta y \\ 0 & D(e^{\alpha x}\cos\beta y) \end{vmatrix}$

$= \begin{vmatrix} 1 & e^{\alpha x}\cos\beta y \\ 0 & \alpha e^{\alpha x}\cos\beta y - \beta e^{\alpha x}\sin\beta y \end{vmatrix}$

$= \alpha e^{\alpha x}\cos\beta y - \beta e^{\alpha x}\sin\beta y - 0$

$= \alpha e^{\alpha x}\cos\beta y - \beta e^{\alpha x}\sin\beta y$

$= D(\phi)$

Hence, $\overline{W}[1, \phi] = D(\phi)$

<u>**Verification of property**</u> <u>5.5</u>:

Let $\phi = e^x\cos y$ and $= e^x\sin y$ and let α be a contants.

Now, $\overline{W}[\phi + \alpha, \boldsymbol{\psi} + \alpha]$

$= \begin{vmatrix} e^x\cos y + \alpha & e^x\sin y + \alpha \\ D(e^x\cos y + \alpha) & D(e^x\sin y + \alpha) \end{vmatrix}$

$= (e^x\cos y + \alpha)(e^x\sin y + e^x\cos y) - (e^x\sin y + \alpha)(e^x\cos y - e^x\sin y)$

$= e^{2x} + 2\alpha e^x\sin y$ (10)

Now, $\overline{W}[\phi, \boldsymbol{\psi}] = \begin{vmatrix} e^x\cos y & e^x\sin y \\ D(e^x\cos y) & D(e^x\sin y) \end{vmatrix}$

$= e^x\cos y\,(e^x\sin y + e^x\cos y) - e^x\sin y\,(e^x\cos y - e^x\sin y)$

$= e^{2x}$ (11)

And, $\alpha D[\phi - \boldsymbol{\psi}] = \alpha\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)[e^x\cos y - e^x\sin y]$

$= \alpha\,[e^x\sin y + e^x\cos y - (e^x\cos y - e^x\sin y)]$

$=2\alpha e^{x}\text{siny}$ (12)

Adding, (11) and (12), we have,

$\bar{W}$ [ϕ, **ψ**] + α D [ϕ - **ψ**] = e^{2x} + $2\alpha e^{x}$siny (13)

From (10) and (13) we conclude that,

$\bar{W}$ [ϕ + α, **ψ** + α] = $\bar{W}$ [ϕ, **ψ**] + α D [ϕ - **ψ**]

<u>**Verification of property** 5.6</u>:

Let $\phi_1 = \cos(x+y)$ and $\phi_2 = \sin(x+y)$

Now, $\bar{W}[\frac{\phi_2}{\phi_1}, \phi_1] = \bar{W}[\frac{\sin(x+y)}{\cos(x+y)}, \cos(x+y)]$

$$= \begin{vmatrix} \frac{\sin(x+y)}{\cos(x+y)} & \cos(x+y) \\ D(\frac{\sin(x+y)}{\cos(x+y)}) & D(\cos(x+y)) \end{vmatrix}$$, where, $D(\phi) = (\frac{\partial}{\partial x} + \frac{\partial}{\partial y})(\phi)$

$$= \begin{vmatrix} \frac{\sin(x+y)}{\cos(x+y)} & \cos(x+y) \\ \frac{2}{\cos^2(x+y)} & -2\sin(x+y) \end{vmatrix} = \frac{-2(1+\sin^2(x+y)}{\cos(x+y)} \quad (14)$$

And, $\bar{W}[\frac{\phi_1}{\phi_2}, \phi_2] = \bar{W}[\frac{\cos(x+y)}{\sin(x+y)}, \sin(x+y)]$

$$= \begin{vmatrix} \frac{\cos(x+y)}{\sin(x+y)} & \sin(x+y) \\ D(\frac{\cos(x+y)}{\sin(x+y)}) & D(\sin(x+y)) \end{vmatrix} = \frac{2(1+\cos^2(x+y)}{\sin(x+y)} \quad (15)$$

Now using (14) and (15) we get,

$\phi_1 \bar{W}[\frac{\phi_2}{\phi_1}, \phi_1] + \phi_2 \bar{W}[\frac{\phi_1}{\phi_2}, \phi_2]$

$$= \cos(x+y)\{\frac{-2(1+\sin^2(x+y)}{\cos(x+y)}\} + \sin(x+y)\{\frac{2(1+\cos^2(x+y)}{\sin(x+y)}\}$$

= 2 cos 2(x + y) (16)

$D(\phi_1\phi_2) = (\frac{\partial}{\partial x} + \frac{\partial}{\partial y})[\cos(x+y)\sin(x+y)]$

= 2 cos 2(x + y) (17)

From (16) and (17) we conclude that,

$\phi_1 \bar{W}[\frac{\phi_2}{\phi_1}, \phi_1] + \phi_2 \bar{W}[\frac{\phi_1}{\phi_2}, \phi_2] = D(\phi_1\phi_2)$

6. Derivative of Partial Wronskian:

If two functions ϕ (x, y) and **ψ**(x, y) which are partial differentiable with respect to x and y on an region R linearly dependent on R, then

D $\bar{W}$ [ϕ, **ψ**] = (ϕ ψ_{xx} − **ψ** ϕ_{xx}) + 2 (ϕ ψ_{xy}- ψϕ_{xx}) + (ϕ ψ_{yy} - ψϕ_{yy})

Some properties:

6.1. D $\bar{W}$ [ϕ, **ψ**] = - D $\bar{W}$ [**ψ**, ϕ]
6.2. D $\bar{W}$ [ϕ + α, **ψ** + α] = D$\bar{W}$ [ϕ, **ψ**] + α D^2 [ϕ - **ψ**]
6.3. D $\bar{W}$[α ϕ, β **ψ**] = $\alpha\beta$ D $\bar{W}$[ϕ, **ψ**]
6.4. $\bar{W}$[1, ϕ] = D^2 (ϕ)

7. Theorem:

If two functions ϕ (x, y) and **ψ**(x, y) which are partial differentiable with respect to x and y are the solution of a partial differential equation of the form,

L (z(x, y)) = P(z_{xx} - 2z_{xy} + z_{yy}) + Q (z_x - z_y) + R (z(x, y)) = 0 (18)

on a region R. Then, $\bar{W}$ [ϕ, **ψ**] satisfied the partial differential equation

$$z_x - z_y + \frac{Q}{P} z = 0$$

that is, $\frac{\partial \bar{W}}{\partial x} - \frac{\partial \bar{W}}{\partial y} + \frac{Q}{P}\bar{W} = 0.$

The functions ϕ (x, y) and **ψ**(x, y) also satisfies the equation, ϕ **L**(**ψ**) – **ψ** L (ϕ) = 0

Verification of theorem 7:

Let us consider a partial differential equation of the form,

$$r - 2s + t - 3p + 3q + 2z = 0 \quad (19)$$

The pde (15) can be written as, (r - 2s + t) - 3(p - q) + 2 z = 0 (20)

Comparing (18) and (20) we have, P = 1, Q = - 3 and R = 1

The fundamental solutions of partial differential equation (19) is,

z = e^x ϕ_1(y + x) + $e^{2x}\phi_2$ (y + x), where ϕ_1 and ϕ_1 are being arbitrary function.

Let, ϕ (x, y) = e^x(y + x) and **ψ**(x, y) = e^{2x}(y + x)

So, $\bar{W}$ [ϕ, **ψ**] ≠ 0

Now, $\bar{W}$ [ϕ, **ψ**] = $\begin{vmatrix} \phi & \psi \\ D(\phi) & D(\psi) \end{vmatrix}$, where, D = $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)$

$$= \begin{vmatrix} e^x(y + x) & e^{2x}(y + x) \\ e^x(y + x + 2) & e^{2x}(2y + 2x + 2) \end{vmatrix}$$

$$= e^{3x}(x^2 + y^2 + 2xy)$$

So, $\bar{W}$ [ϕ, **ψ**] = $e^{3x}(x^2 + y^2 + 2xy)$

Now, $$\frac{\partial \bar{W}}{\partial x} = e^{3x}(3x^2 + 3y^2 + 6xy + 2x + 2y) \quad (21)$$

$$\frac{\partial \bar{W}}{\partial y} = e^{3x}(2y + 2x) \quad (22)$$

$$\frac{Q}{P}\bar{W} = \frac{-3}{1} e^{3x}(x^2 + y^2 + 2xy) \quad (23)$$

From, (21), (22) and (23), we have,

$$\frac{\partial \bar{W}}{\partial x} - \frac{\partial \bar{W}}{\partial y} + \frac{Q}{P}\bar{W} = 0. \quad (24)$$

Now,

$$L(\psi) = \psi_{xx} - 2\psi_{xy} + \psi_{yy} - 3\psi_x + 3\psi_y + 2\psi = 0 \quad (25)$$

$$L(\phi) = \phi_{xx} - 2\phi_{xy} + \phi_{yy} - 3\phi_x + 3\phi_y + 2\phi = 0 \quad (26)$$

Now, from (25) and (26) we conclude that,

ϕ L (**ψ**) – **ψ** L (ϕ) = 0

8. Conclusion:

Let us emphasize in brief the results of this work. In our research work we reviewed many research paper and we observed that the application of Wronskian is limited in the field of partial differential equation. In this paper we discuss about the properties and fundamental solutions of partial differential equations. We verified some of the properties of Partial Wronskian. We have discussed a theorem (8) and verified the result (24) using a linear partial differential equation.

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Reference:

[1]. LIU Jin, DENG Shu- Fang (2003), Novel Wronskian solution of the Kdv Equation. Journal of Shangai University, Volume 7 (3): 223 – 227

[2]. K Ravi, B. Ravikrishnan (2006) solution of Wronskian differential equation. International Journal of Pure and Applied Mathematics, 27, No. 2, 263-273

[3]. K Ravi, B. Ravikrishnan (2007) Wronskian differential equations, Applied Science Periodical IX No 1

[4]. G. Ganapathy, S Murthy, M Arun Kumar, (2010) Properties of Wronskian and Partial Wronskian, International Journal of pure and Applied Mathematics, Volume 63, No1 (21 – 30)

[5] Ram Shankar Subramaniam (26 June 2020), Solution of Partial Differential Equations: Separation of variables.

[6]. Mr. Ikbal Hussain Laskar, Dr. Gitu Mani Sarma, (2020) some discussion of properties of Wronskian in boundary value problem, International Journal of Mathematics Trends and Technology, volume 66 Issue 1.

[7]. David J. B Llod (4th feb 2015), Localized solution of Partial differential equation.