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Applications Of Analytic Functions Related To Mittag-Leffler Function

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ABSTRACT
For analytic function f in the open unit disc E, a linear operator defined by Mittag-Leffler function is introduced. The object of the present paper is to study some properties for $D_\lambda^m(\upsilon,\tau)f(z)$ belonging to some classes by applying the concept of Jack's lemma. Subordination relations are introduced.

Keywords and phrases: : analytic function, starlike, differential operator, Mittag-Leffler function, Jack's lemma.

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1 INTRODUCTION

Let A denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n , \qquad (1.1)$$

in the open the unit disk $E = \{z : |z| < 1\}$. Let S be the subclass of **A** consisting of univalent functions and satisfy the following usual normaliza-tion condition $f(0) = 0$ and $f'(0) = 1$. We denote by S the subclass of **A** consisting of $f(z)$ which are all univalent in E. A function $f \in$ **A** is a starlike function of the order υ, $\upsilon(0 \le \upsilon < 1)$ if it satisfy

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \upsilon, \quad (z \in E), \qquad (1.2)$$

we denote by this class $S^*(\upsilon)$. A function $f \in$ **A** is a convex function of the order υ , $\upsilon\ (0 \le \upsilon < 1)$ if it satisfy

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \upsilon, \quad (z \in E) \qquad (1.3)$$

we denote this class with $\boldsymbol{K}(\upsilon)$. For $f \in$ **A** given by (1.1) and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \qquad (1.4)$$

their convolution (or Hadamard product), denoted by $(f * g)$, is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z), \quad (z \in E) \qquad (1.5)$$

Note that $f * g \in$ **A.** The following defines the familiar Mittag-Leffler func-tion $E_\upsilon(z)$ introduced by Mittag- Leffler [4] and its generalization $E_{\upsilon,\tau}(z)$ introduced by Wiman [9].

$$E_v(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(vn+1)}$$

and

$$E_{v,\tau}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(vn+\tau)}$$

where v, $\tau \in$ C, $\Re(\upsilon) > 0$ and $\Re(\tau) > 0$. We define the function $Q_{u,\tau}(z)$ by $Q_{v,\tau}(z) = z\Gamma(\tau)\, E_{v,\tau}(z)$.

Now, for $f \in$ A, we define the following differential operator $D_\lambda^m(\upsilon,\tau) f : A \to A$ by

$$D_\lambda^0(\upsilon,\tau)f(z) = f(z) * Q_{v,\tau}(z)$$

$$D_\lambda^1(\upsilon,\tau)f(z) = (1-\lambda)(f(z) * Q_{v,\tau}(z)) + \lambda z(f(z) * Q_{v,\tau}(z))'$$

$$\vdots$$

$$D_\lambda^m(\upsilon,\tau)f(z) = D_\lambda^1(D_\lambda^{m-1}(v,\tau)f(z)).$$

If f is given by (1.1) then from the definition of the operator $D_\lambda^m f$ it is easyto see that

$$D_\lambda^m(\upsilon,\tau) f(z) = z + \sum_{n=2}^{\infty} \emptyset_n^m(\lambda,\upsilon,\tau) a_n z^n, \quad (1.6)$$

where

$$\phi_n^m(\lambda,\upsilon,\tau) = \frac{\Gamma(\tau)}{\Gamma(\upsilon(n-1)+\tau)}[\lambda(n-1)+1]^m. \quad (1.7)$$

Note that

1. when $\upsilon = 0$ and $\tau = 1$, we get Al-Oboudi operator [1].
2. when $\upsilon = 0$, $\tau = 1$ and $\lambda = 1$, we get Salagean operator [7].
3. when m = 0, we get $E_{\upsilon,\tau}(z)$, Srivastava et al. [8].

Now, by making use of the differential operator $D_\lambda^m(\upsilon,\tau) f$, we define a new subclass of functions belonging to the class A.

Definition 1.1. Let a function f ∈ A. Then $f \in D_\lambda^m(\upsilon,\tau) f(z)$ if and only if

$$\Re\left\{\frac{z\left(D_\lambda^m(\upsilon,\tau) f(z)\right)'}{D_\lambda^m(\upsilon,\tau) f(z)}\right\} > \varrho, \quad z \in E, 0 \le \varrho \le 1. \quad (1.8)$$

Let f and g be analytic in E. Then f is said to be subordinate to g if there exists an analytic function ω satisfying $\omega(0) = 0$ and $\omega(z) < 1$, such that $f(z) = g(\omega z)$, $z \in E$. We denote this subordination as $f(z) \prec g(z)$ or ($f \prec g$), $z \in E$. The basic idea in proving our result is the following lemma due to Jack [2] (also, due to Miller and Mocannu [3]).

Lemma 1.2. Let $\omega(z)$ be analytic in E with $\omega(0) = 0$. Then if $|\omega(z)|$ attains its maximum value on the circle $|z| = r$ at a point z_0 in E then we have $z_0\,\omega'(z) = k\,\omega(z_0)$, where $k \ge 1$ is a real number.

2 MAIN RESULTS

In the present paper, we follow similar works done by Shireishi and Owa [6] and Ochiai et al. [5], we derive the following result.

Theorem 2.1. If f ∈ A satisfies $\Re\left\{\frac{z\left(D_\lambda^m(\upsilon,\tau) f(z)\right)'}{D_\lambda^m(\upsilon,\tau) f(z)}\right\} < \frac{\varrho-3}{2(\varrho-1)}$, $z \in E$ for some ϱ $(-1 < \varrho \le 0)$ then $\frac{D_\lambda^m(\upsilon,\tau) f(z)}{z} \prec \frac{1+\varrho z}{1-z}$, $z \in E$.

This implies that

$$\Re\left\{\frac{D_\lambda^m(\upsilon,\tau) f(z)}{z}\right\} > \frac{1-\varrho}{2}.$$

Proof. Let us define the function $\omega(z)$ by $\frac{D_\lambda^m(\upsilon,\tau) f(z)}{z} = \frac{1-\varrho\omega(z)}{1-\omega(z)}, (\omega(z) \neq 1)$.

Clearly, $\omega(z)$ is analytic in E and $\omega(0) = 0$. We want to prove that $|\omega(z)| < 1$ in E. Since

$$\frac{z(D_\lambda^m(\upsilon,\tau) f(z))'}{D_\lambda^m(\upsilon,\tau) f(z)} = \frac{-\varrho z\omega'(z)}{1-\varrho\omega(z)} + \frac{z\omega'(z)}{1-\omega(z)} + 1,$$

we see that

$$\Re\left\{\frac{z\left(D_\lambda^m(\upsilon,\tau) f(z)\right)'}{D_\lambda^m(\upsilon,\tau) f(z)}\right\} = \Re\left\{\frac{-\varrho z\omega'(z)}{1-\varrho\omega(z)} + \frac{z\omega'(z)}{1-\omega(z)} + 1\right\} < \frac{\varrho-3}{2(\varrho-1)}, (z \in E)$$

for $-1 < \varrho \le 0$. If there exists a point $z_0 \in E$ such that $\max_{|z|\le|z_0|} |\omega(z)| = |\omega(z_0)| = 1$, then Lemma 1.2, gives us that $\omega(z_0) = e^{i\theta}$ and $z_0\omega'(z_0) = k\omega(z_0)$, $k \ge 1$.

Thus we have

$$\frac{z_0(D_\lambda^m(\upsilon,\tau) f(z_0))'}{D_\lambda^m(\upsilon,\tau) f(z_0)} = \frac{-\varrho z_0\omega'(z_0)}{1-\varrho\omega(z_0)} + \frac{z_0\omega'(z_0)}{1-\omega(z_0)} + 1 = 1 + \frac{k}{1-e^{i\theta}} - \frac{k}{1-\varrho e^{i\theta}}.$$

It follows that

$$\Re\left\{\frac{1}{1-\omega(z_0)}\right\} = \Re\left\{\frac{1}{1-e^{i\theta}}\right\} = \frac{1}{2}$$

and $\Re\left\{\frac{1}{1-\omega(z_0)}\right\} = \Re\left\{\frac{1}{1-\varrho e^{i\theta}}\right\} = \frac{1}{2} - \frac{1-\varrho^2}{2(1+\varrho^2-2\varrho\cos\theta)}$.

Therefore, we have

$$\Re\left\{\frac{z_0(D_\lambda^m(\upsilon,\tau) f(z_0))'}{D_\lambda^m(\upsilon,\tau) f(z_0)}\right\} = 1 - \frac{k(\varrho^2-1)}{2(1+\varrho^2-2\varrho\cos\theta)}.$$

This implies that $-1 < \varrho \le 0$, $\Re\left\{\frac{z_0(D_\lambda^m(\upsilon,\tau) f(z_0))'}{D_\lambda^m(\upsilon,\tau) f(z_0)}\right\} \ge 1 + \frac{(1-\varrho^2)}{2(\varrho-1)^2} = \frac{\varrho-3}{2(\varrho-1)}$.

This contradicts the condition in the theorem. Then there is no $z_0 \in E$ such that $|\omega(z_0)| = 1$ for all z ∈ E, that is $\left(\frac{D_\lambda^m(\upsilon,\tau) f(z)}{z}\right) \prec \frac{1+\varrho z}{1-z}$, $z \in E$.

Further more, since

$$\omega(z) = \frac{\frac{D_\lambda^m(\upsilon,\tau) f(z)}{z} - 1}{\frac{D_\lambda^m(\upsilon,\tau) f(z)}{z} - \varrho}, \quad z \in E.$$

and $|\omega(z)| < 1$, $(z \in E)$, we conclude that $\Re\left\{\frac{D_\lambda^m(\upsilon,\tau) f(z)}{z}\right\} > \frac{1-\varrho}{2}$.

Taking $\varrho = 0$ in the Theorem 2.1, we have the following corollary.

Corollary 2.2. If f ∈ A satisfies

$$\Re\left\{\frac{z\left(D_\lambda^m(\upsilon,\tau) f(z)\right)'}{D_\lambda^m(\upsilon,\tau) f(z)}\right\} > \frac{3}{2}, \quad z \in E$$

then

$$\frac{D_\lambda^m(\upsilon,\tau)f(z)}{z} \prec \frac{1}{1-z}, \qquad z \in E$$

and

$$\Re\left\{\frac{D_\lambda^m(\upsilon,\tau)f(z)}{z}\right\} > \frac{1}{2}, \qquad z \in E$$

Theorem 2.3. If $f \in A$ satisfies

$$\Re\left\{\frac{z\left(D_\lambda^m(\upsilon,\tau)f(z)\right)'}{D_\lambda^m(\upsilon,\tau)f(z)}\right\} > \frac{3\varrho-1}{2(\varrho-1)}, \qquad z \in E$$

for some ϱ $(-1 < \varrho \le 0)$ then $\frac{z}{D_\lambda^m(\upsilon,\tau)f(z)} \prec \frac{1+z}{1-z}$, $z \in E$

and

$$\left|\frac{D_\lambda^m(\upsilon,\tau)f(z)}{z} - \frac{1}{1-\varrho}\right| < \frac{1}{1-\varrho}, \qquad z \in E.$$

This implies that $\Re\left\{\frac{D_\lambda^m(\upsilon,\tau)f(z)}{z}\right\} > 0,\ z \in E$.

Proof. Let us define the function $\omega(z)$ by

$$\frac{z}{D_\lambda^m(\upsilon,\tau)f(z)} = \frac{1-\varrho\omega(z)}{1-\omega(z)},\ (\omega(z) \neq 1). \qquad (2.1)$$

Then, we have $\omega(z)$ is analytic in E and $\omega(0) = 0$. We want to prove that $|\omega(z)| < 1$ in E. Differentiating equation (2.1), we obtain

$$\frac{z(D_\lambda^m(\upsilon,\tau)f(z))'}{D_\lambda^m(\upsilon,\tau)f(z)} = \frac{-z\omega'(z)}{1-\omega(z)} + \frac{\varrho z\omega'(z)}{1-\varrho\omega(z)} + 1,$$

$$\Rightarrow \Re\left\{\frac{z\left(D_\lambda^m(\upsilon,\tau)f(z)\right)'}{D_\lambda^m(\upsilon,\tau)f(z)}\right\} = \Re\left\{\frac{-z\omega'(z)}{1-\omega(z)} + \frac{\varrho z\omega'(z)}{1-\varrho\omega(z)} + 1\right\} > \frac{3\varrho-1}{2(\varrho-1)}, \qquad (z \in E),$$

for $(-1 < \varrho \le 0)$. If there exists a point $(z_0 \in E)$ such that Lemma 1.2, gives us that $\omega(z_0) = e^{i\theta}$ and $z_0\omega'(z_0) = k\omega(z_0)$, $k \ge 1$. Thus we have

$$\frac{z_0(D_\lambda^m(\upsilon,\tau)f(z_0))'}{D_\lambda^m(\upsilon,\tau)f(z_0)} = \frac{-z_0\omega'(z_0)}{1-\omega(z_0)} + \frac{\varrho z_0\omega'(z_0)}{1-\varrho\omega(z_0)} + 1 = 1 + \frac{k}{1-e^{i\theta}} - \frac{k}{1-\varrho e^{i\theta}}$$

Therefore, we have

$$\Re\left\{\frac{z_0\left(D_\lambda^m(\upsilon,\tau)f(z_0)\right)'}{D_\lambda^m(\upsilon,\tau)f(z_0)}\right\} = 1 + \frac{k(\varrho^2-1)}{2(1+\varrho^2-2\varrho\cos\theta)}.$$

This implies that, for $-1 < \varrho \le 0$,

$$\Re\left\{\frac{z_0\left(D_\lambda^m(\upsilon,\tau)f(z_0)\right)'}{D_\lambda^m(\upsilon,\tau)f(z_0)}\right\} = 1 - \frac{k(1-\varrho^2)}{2(1+\varrho^2-2\varrho\cos\theta)} \le \frac{3\varrho-1}{2(\varrho-1)}.$$

This contradicts the condition in the theorem. Hence, there is no $z_0 \in E$ such that $|\omega(z_0)| = 1$ for all $z \in E$, that is

$$\frac{z}{D_\lambda^m(\upsilon,\tau)f(z)} \prec \frac{1+z}{1-z}, \qquad z \in E.$$

Furthermore, since $\omega(z) = \frac{1-\frac{D_\lambda^m(\upsilon,\tau)f(z)}{z}}{1-\frac{\varrho D_\lambda^m(\upsilon,\tau)f(z)}{z}},\ z \in E$

and $|\omega(z)| < 1$, $(z \in E)$ we conclude that

$$\left|\frac{D_\lambda^m(\upsilon,\tau)f(z)}{z} - \frac{1}{1-\varrho}\right| < \frac{1}{1-\varrho}, \qquad z \in E$$

which implies that $\Re\left\{\frac{D_\lambda^m(\upsilon,\tau)f(z)}{z}\right\} > 0,\ z \in E$.

We complete the proof of the theorem.

By setting $\varrho = 0$ in Theorem 2.3, we readily obtain the following.

Corollary 2.4. If $f \in A$ satisfies

$$\Re\left\{\frac{z(D_\lambda^m(\upsilon,\tau)f(z))'}{D_\lambda^m(\upsilon,\tau)f(z)}\right\} > \frac{1}{2}, \qquad z \in E.$$

then

$$\frac{z}{D_\lambda^m(\upsilon,\tau)f(z)} \prec \frac{1+z}{1-z}, \qquad z \in E.$$

and

$$\left|\frac{D_\lambda^m(\upsilon,\tau)f(z)}{z} - 1\right| < 1, \qquad z \in E$$

Theorem 2.5. If $f \in A$ satisfies

$$\Re\left\{\frac{z(D_\lambda^m(\upsilon,\tau)f(z))'}{D_\lambda^m(\upsilon,\tau)f(z)}\right\} < \frac{\varrho(2-\gamma)-(2+\gamma)}{2(\varrho-1)}, \qquad z \in E.$$

for some ϱ $(-1 < \varrho \le 0)$ and $0 < \gamma \le 1$ then

$$\left(\frac{D_\lambda^m(\upsilon,\tau)f(z)}{z}\right)^{\frac{1}{\gamma}} \prec \frac{1+\varrho z}{1-z}, \qquad z \in E.$$

This implies that

$$\Re\left(\left(\frac{D_\lambda^m(\upsilon,\tau)f(z)}{z}\right)^{\frac{1}{\gamma}}\right) > \frac{1-\varrho}{2}, \qquad z \in E.$$

Proof. Let us define the function $\omega(z)$ by

$$\frac{D_\lambda^m(\upsilon,\tau)f(z)}{z} = \left(\frac{1-\varrho\omega(z)}{1-\omega(z)}\right)^{\gamma}, \omega(z) \neq 1.$$

Clearly, $\omega(z)$ is analytic in E and $\omega(0)) = 0$. We want to prove that $|\omega(z)| < 1$ in E. Since

$$\frac{z(D_\lambda^m(\upsilon,\tau)\, f(z))'}{D_\lambda^m(\upsilon,\tau)\, f(z)} = \gamma\left(\frac{z\omega'(z)}{1-\omega(z)} - \frac{\varrho z\omega'(z)}{1-\varrho\omega(z)}\right) + 1.$$

We see that

$$\Re\left\{\frac{z(D_\lambda^m(\upsilon,\tau)\, f(z))'}{D_\lambda^m(\upsilon,\tau)\, f(z)}\right\} = \Re\left\{\gamma\left(\frac{z\omega'(z)}{1-\omega(z)} - \frac{\varrho z\omega'(z)}{1-\varrho\omega(z)}\right) + 1\right\} < \frac{\varrho(2-\gamma)-(2+\gamma)}{2(\varrho-1)}, \qquad z \in E,$$

for ϱ $(-1 < \varrho \leq 0)$ and $0 < \gamma \leq 1$. If there exists a point $(z_0 \in E)$ such that $\max_{|z|\leq|z_0|}|\omega(z)| = |\omega(z_0)| = 1$,
then by Lemma 1.2, gives us that $\omega(z_0) = e^{i\theta}$ and $z_0\omega'(z_0) = k\omega(z_0)$, $k \geq 1$.

Thus we have

$$\frac{z_0(D_\lambda^m(\upsilon,\tau)\, f(z_0))'}{D_\lambda^m(\upsilon,\tau)\, f(z_0)} = \gamma\left(\frac{z_0\omega'(z_0)}{1-\omega(z_0)} - \frac{\varrho z_0\omega'(z_0)}{1-\varrho\omega(z_0)}\right) + 1 = 1 + \frac{k}{1-e^{i\theta}} - \frac{k}{1-\varrho e^{i\theta}}.$$

Therefore, we have

$$\Re\left\{\frac{z_0(D_\lambda^m(\upsilon,\tau)\, f(z_0))'}{D_\lambda^m(\upsilon,\tau)\, f(z_0)}\right\} = 1 + \frac{\gamma k(1-\varrho^2)}{2(1+\varrho^2-2\varrho\cos\theta)}.$$

Thus implies that, for ϱ $(-1 < \varrho \leq 0)$ and $0 < \gamma \leq 1$

$$\Re\left\{\frac{z_0(D_\lambda^m(\upsilon,\tau)\, f(z_0))'}{D_\lambda^m(\upsilon,\tau)\, f(z_0)}\right\} \geq \frac{\varrho(2-\gamma)-(2+\gamma)}{2(\varrho-1)}.$$

This contradicts the condition in the theorem. Hence, there is no $z_0 \in E$ such that $|\omega(z_0)| = 1$ for all $z \in E$, that is

$$\left(\frac{D_\lambda^m(\upsilon,\tau)\, f(z)}{z}\right)^{\frac{1}{\gamma}} \prec \frac{1-\varrho z}{1-z}, \qquad z \in E.$$

Furthermore, since

$$\omega(z) = \frac{\left(\frac{D_\lambda^m(\upsilon,\tau)\, f(z)}{z}\right)^{\frac{1}{\gamma}} - 1}{\left(\frac{D_\lambda^m(\upsilon,\tau)\, f(z)}{z}\right)^{\frac{1}{\gamma}} - \varrho}$$

and $|\omega(z)| < 1$, $(z \in E)$, we conclude that

$$\Re\left(\frac{D_\lambda^m(\upsilon,\tau)\, f(z)}{z}\right)^{\frac{1}{\gamma}} > \frac{1-\varrho}{2}, z \in E,$$

we complete the proof of the theorem.

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