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On A Subclass Analytic Functions Involving Gegenbauer Polynomials

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ABSTRACT:
The target of this paper is to study a new subclass of analytic functions defined by a Gegenbauer polynomial and obtained coefficient estimates, growth and distortion theorems, radii of starlikeness, convexity and close-to-convexity are obtained. Furthermore, we otained integral means inequalities for the function.

Keywords: analytic, coefficient bounds, starlike, distortion.

2010 Mathematics subject classification: 30C45

1. INTRODUCTION

Let A denote the class of functions *f* of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

which are analytic in the open unit disk $E = \{z \in C : |z| < 1\}$.

A function *f* in the class A is said to be in the class $ST(\alpha)$ of starlike functions of order α in E, if it satisfies the inequality

$$Re\{\frac{zf'(z)}{f(z)}\} > a, \quad (0 \leq \alpha < 1), \quad (z \in E) \qquad (1.2)$$

Note that is the class of starlike functions.

Denote by T the subclass of A consisting of functions *f* of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0) \qquad (1.3)$$

This subclass was introduced and extensively studied by Silverman [6].

For $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ the Hadamard product of *f* and *g* is defined by $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad (z \in E)$.
Orthogonal polynomials have been widely studied in recent years from various perspective due to their importance in mathematical physics, mathematical statistics, engineering, and probability theory.
Orthogonal polynomials that appear most commonly in applications are the classical orthogonal polynomials (Hermite polynomials, Laguerre polynomials and Jacobi polynomials). The general subclass of Jacobi polynomials is the set of Gegenbauer polynomials, this class includes Legendre polynomials and Chebyshev polynomials as subclasses.
Orthogonal polynomials have been studied extensively as early as they key were discovered by Legendre in 1784[7].
In the mathematical treatment of model problems, orthogonal polynomials arise often to find solutions of ordinary differential equations under certain conditions imposed by the model.
Th importance of the orthogonal polynomials for the contemporary mathematics, as well as for wide range of their applications in the physics and engineering is beyond any doubt. It is well-known that these polynomials play an essential role in problems of the approximation theory. They occur in the theory of differential and integral equations as well as in the mathematical statistics. Their applications in the quantum mechanics, scattering theory, automatic control, signal analysis and axially symmetric potential theory are also known [2,4].

Finally speaking, polynomials P_n and P_m of order n and m are orthogonal if $\int_a^b w(x)P_n(x)P_m(x)dx = 0$, for $n \neq m$.
where w(x) is non-negative function in the interval (a,b); therefore, the integral is well defined for all finite order polynomials $P_n(x)$.
A special case of orthogonal polynomials are Gegenbauer polynomials. They are representatively related with typically real functions and generating function of Gegenbauer polynomials are using common algebraic expressions. Undoubtedly, this led to several useful inequalities appear from the Gegenbauer polynomials realm.
Typically real function play an important role in geometric function theory because of the relation $T_R = coS_R$ and its role of estimating coefficients bound, where S_R denotes the class of univalent functions in the unit disk with real coefficients,and coS_R denotes the closed convex hull of S_R.
Gegenbauer polynomials are extensively studied by various authors see [1,5,16,18,19]. The class $T(\lambda), \lambda \geq 0$ were introduced and investigated by Szynal [17] as the subclass of A consisting of functions of the form

$$f(z) = \int_{-1}^{1} k(z,m)d\mu(m) \quad (1.4)$$

where

$$k(z,m) = \frac{z}{(1-2mz+z^2)^\lambda} m \in [-1,1], (z \in E) \quad (1.5).$$

And μ is a probability measure on the interval [-1,1].
The collection of such measure on [a,b] is denoted by $P_{[a,b]}$.
The Taylor series expansion of the function in (1.5) gives

$$k(z,t) = z + c_1^\lambda(m)z^2 + c_2^\lambda z^3 + \cdots \quad (1.6)$$

and the coefficients for (1.6) were given below:

$$c_0^\lambda(m) = 1, c_1^\lambda(m) = 2\lambda m, c_2^\lambda(m) = 2\lambda(\lambda+1)(m)^2 - \lambda, c_3^\lambda(m) = \frac{4}{3}\lambda(\lambda+1)(\lambda+2)m^3 - 2\lambda(\lambda+1)m, \ldots$$

Where $c^\lambda{}_n(m)$ denotes the Gegenbauer polynomial of degree n. Varying the parameter λ in (1.6). We can obtain the class of typically real functions studied by [3],[10],[11] and [15].
Let $G_\lambda^m f(z) : A \to A$ defined in terms of the convolution by $G_\lambda^m f(z) = k(z,m) * f(z)$, We have $G_\lambda^m f(z) = z + \sum_{n=2}^{\infty} \phi_n(\lambda,m)a_n z^n$, (1.8)

Where $\phi_n(\lambda,m) = c_{n-1}^\lambda(m)$.
In this paper, using the operator $\mathscr{G}_\lambda^m f(z)$, we define the following new class motivated by Murugusunder amoorthy and Magesh[9].

Definition 1: The function *f(z)* of the form (1.1) is in the class $S_\lambda^m(\mu,\gamma)$ if it satisfies the inequality

$$\text{Re}\left\{\frac{z(\mathscr{G}_\lambda^m f(z))}{(1-\mu)z+\mu\mathscr{G}_\lambda^m f(z)} - \gamma\right\} > \left|\frac{z(\mathscr{G}_\lambda^m f(z))}{(1-\mu)z+\mu\mathscr{G}_\lambda^m f(z)} - 1\right|.$$

for $0 \leq \lambda \leq 0$, and $0 \leq \gamma < 1$.
Further we define $TS_\lambda^m(\mu,\gamma) = S_\lambda^m(\mu,\gamma) \cap T$.
The aim of this paper is to study the coefficient bounds, radii of close-to-convex and starlikeness convex linear combination for the class $TS_\lambda^m(\mu,\gamma)$. Furthermore, we obtained integral means inequalities for the function in $TS_\lambda^m(\mu,\gamma)$.

Theorem 1: A function *f(z)* of the form (1.1) is in $S_\lambda^m(\mu,\gamma)$.

$$\sum_{n=2}^{\infty}[2n-\mu(\gamma+1)]\phi_n(\gamma,m)|a_n| \leq 1-\gamma.$$

where , $0 \leq \mu \leq 1$ and $0 \leq \gamma < 1$, and $\phi_n(\lambda,m)$ is given by (1.8)

Proof: It suffices to show that

$$\left|\frac{z(\mathscr{G}_\lambda^m f(z))}{(1-\mu)z+\mu\mathscr{G}_\lambda^m f(z)} - 1\right| - \text{Re}\left\{\frac{z(\mathscr{G}_\lambda^m f(z))}{(1-\mu)z+\mu\mathscr{G}_\lambda^m f(z)} - 1\right\} \leq 1-\gamma$$

We have

$$\left|\frac{z(\mathscr{G}_\lambda^m f(z))}{(1-\mu)z+\mu\mathscr{G}_\lambda^m f(z)} - 1\right| - \text{Re}\left\{\frac{z(\mathscr{G}_\lambda^m f(z))}{(1-\mu)z+\mu\mathscr{G}_\lambda^m f(z)} - 1\right\}$$

$$\leq 2\left|\frac{z(\mathscr{G}_\lambda^m f(z))}{(1-\mu)z+\mu\mathscr{G}_\lambda^m f(z)} - 1\right|$$

$$\leq 2\frac{\sum_{n=2}^{\infty}(n-\mu)\phi_n(\lambda,m)|a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty}\mu\phi_n(\lambda,m)|a_n||z|^{n-1}}$$

$$\leq 2\frac{\sum_{n=2}^{\infty}(n-\mu)\phi_n(\lambda,m)|a_n|}{1-\sum_{n=2}^{\infty}\mu\phi_n(\lambda,m)|a_n|}$$

The last expression is bounded above by $(1-\gamma)$ if $\sum_{n=2}^{\infty}[2n-\mu(\gamma+1)]\phi_n(\lambda,m)|a_n| \leq 1-\gamma$ and the proof is complete.

Theorem 2: Let $0 \leq \mu \leq 1$ and $0 \leq \gamma < 1$ then a function f of the form (1.3) to be in the class $TS_\lambda^m(\mu,\gamma)$ if and only if

$$\sum_{n=2}^{\infty}[2n-\mu(\gamma+1)]\phi_n(\lambda,m) \leq 1-\gamma \quad (2.2)$$

where $\phi_n(\lambda,m)$ are given by (1.5.)

Proof: In the view of Theorem 1, we need only to prove the necessity. If $f \in TS_\lambda^m(\mu,\gamma)$ and z is real then

$$\text{Re}\left\{\frac{1-\sum_{n=2}^{\infty}n\phi_n(\lambda,m)a_n z^{n-1}}{1-\sum_{n=2}^{\infty}\mu\phi_n(\lambda,m)a_n z^{n-1}} - \gamma\right\} > \zeta\left|\frac{\sum_{n=2}^{\infty}(n-\mu)\phi_n(\lambda,m)a_n z^{n-1}}{1-\sum_{n=2}^{\infty}\mu\phi_n(\lambda,m)a_n z^{n-1}}\right|$$

Letting z→1 along the real axis, we obtain the desired inequality

$$\sum_{n=2}^{\infty}[2n-\mu(\gamma+1)]\phi_n(\lambda,m)|a_n| \le 1-\gamma.$$

where $0 \le \mu \le 1, 0 \le \gamma < 1$, and $\phi_n(\lambda,m)$ are given by (1.6).

Corollary 1: If $f(z) \in TS_\lambda^m(\mu,\gamma)$ then

$$|a_n| \le \frac{1-\gamma}{[2n-\mu(\gamma+1)]\phi_n(\lambda,m)} \quad (2.3).$$

where $0 \le \mu \le 1, 0 \le \gamma < 1$, and $\phi_n(\lambda,m)$ are given by (1.5). Equality holds for the function

$$f(z) = z - \frac{1-\gamma}{[2n-\mu(\gamma+1)]\phi_n(\lambda,m)}z^n \quad (2.4)$$

Theorem 3: Let $f_1(z)=z$ and

$$f_n(z) = z - \frac{1-\gamma}{[2n-\mu(\gamma+1)]\phi_n(\lambda,m)}z^n, n \le 2. \quad (2.5)$$

Then $f(z) \in TS_\lambda^m(\mu,\gamma)$, if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} w_n f_n(z), w_n \ge 0, \sum_{n=1}^{\infty} w_n = 1 \quad (2.6)$$

Proof: Suppose *f(z)* can be written as in (2.6) then

$$f(z) = z - \sum_{n=2}^{\infty} w_n \frac{1-\gamma}{[2n-\mu(\gamma+1)]\phi_n(\lambda,m)}z^n.$$

Now,

$$\sum_{n=2}^{\infty} w_n \frac{(1-\gamma)[2n-\mu(\gamma+1)]\phi_n(\lambda,m)}{(1-\gamma)[2n-\mu(\gamma+1)]\phi_n(\lambda,m)} = \sum_{n=2}^{\infty} w_n = 1 - w_1 \le$$

Thus $f(z) \in TS_\lambda^m(\mu,\gamma)$.

Conversely, let us have $f(z) \in TS_\lambda^m(\mu,\gamma)$. Then by using (2.3), we get

$$w_n = \frac{[2n-\mu(\gamma+1)]\phi_n(\lambda,m)}{(1-\gamma)}a_n, n \ge 2$$

and $w_1 = 1 - \sum_{n=2}^{\infty} w_n$.

Then we have $f(z) = \sum_{n=1}^{\infty} w_n f_n(z)$ and hence this complete the proof of theorem.

Theorem 4: The class $TS_\lambda^m(\mu,\gamma)$ is a convex set.

Proof: Let the function

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j}z^n, a_{n,j} \ge 0, j = 1,2 \quad (2.7)$$

be in the class $TS_\lambda^m(\mu,\gamma,\zeta)$. It suffices to show that the function *h(z)* defined by

$h(z) = \xi f_1(z) + (1-\xi)f_2(z), 0 \le \xi < 1$,

is in the class $TS_\lambda^m(\mu,\gamma)$. Since

$$h(z) = z - \sum_{n=2}^{\infty}[\xi a_{n,1} + (1-\xi)a_{n,2}]z^n$$

An easy computation with the aid of theorem 2, gives

$\sum_{n=2}^{\infty}[2n-\mu(\gamma+1)]\xi\phi_n(\lambda,m)a_{n,1} + \sum_{n=2}^{\infty}[2n-\mu(\gamma+1)](1-\xi)\phi_n(\lambda,m)a_{n,2}$

$\le \xi(1-\gamma) + (1-\xi)(1-\gamma)$

$\le (1-\gamma)$,

Which implies that $h \in TS_\lambda^m(\mu,\gamma)$.

Hence $TS_\lambda^m(\mu,\gamma)$ is convex.

Next we obtain the radii of close-to-convexity, starlikeness and convexity for the class $TS_\lambda^m(\mu,\gamma)$.

Theorem 5: Let the function *f(z)* defined by (1.3) belong to the class $TS_\lambda^m(\mu,\gamma)$.

Then *f(z)* is close-to-convex of order $\delta(0 \le \delta < 1)$ in the disc $|z| < r_1$, where

$$r_1 = inf_{n\ge2}\left[\frac{(1-\delta)\sum_{n=2}^{\infty}[2n-\mu(\gamma+1)]\phi_n(\lambda,m)}{n(1-\gamma)}\right]^{\frac{1}{n-1}}, n \ge 2 \quad (2.8)$$

The result is sharp, with the extremal function *f(z)* is given by (2.5) **Proof:** Given $f \in T$ and f is close-to-convex of order δ, we have′

$$|f'(z) - 1| < 1 - \delta \quad (2.9)$$

For the left hand side of (2.9) we have

$$|f'(z) - 1| \le \sum_{n=2}^{\infty} na_n|z|^{n-1}$$

The last expression is less than $1 - \delta$

$$\sum_{n=2}^{\infty}\frac{n}{1-\delta}a_n|z|^{n-1} \le 1$$

Using the fact that $f(z) \in TS_\lambda^m(\mu,\gamma)$ if and only if

$$\sum_{n=2}^{\infty}\frac{[2n-\mu(\gamma+1)]\phi_n(\lambda,m)}{(1-\gamma)}a_n \le 1$$

We can (2.9) is true if

$$\frac{n}{1-\delta}|z|^{n-1} \le \frac{(1-\delta)[2n-\mu(\gamma+1)]\phi_n(\lambda,m)}{n(1-\gamma)}$$

Or equivalently,

$$|z| \le \left[\frac{(1-\delta)[2n-\mu(\gamma+1)]\phi_n(\lambda,m)}{n(1-\gamma)}\right]^{\frac{1}{n-1}}$$

Which is complete the proof.

Theorem 6: Let the function *f(z)* defined by (1.3) belong to the class $TS_\lambda^m(\mu,\gamma)$. Then **f(z)** is starlike of order $\delta(0 \le \delta < 1)$ in the disc $|z| < r_2$, where

$$r_2 = inf_{n\ge2}\left[\frac{(1-\delta)\sum_{n=2}^{\infty}[2n-\mu(\gamma+1)]\phi_n(\lambda,m)}{(n-\delta)(1-\gamma)}\right]^{\frac{1}{n-1}} \quad (2.10)$$

The result is sharp, with extremal function *f(z)* is given by (2.5) **Proof:** Given **f** $\in T$ and **f** is starlike of order δ, we have

$$\left[\frac{zf'(z)}{f(z)} - 1\right] < 1 - \delta \quad (2.11)$$

For the left hand side of (2.11) we have

$$\left[\frac{zf'(z)}{f(z)} - 1\right] \le \sum_{n=2}^{\infty}\frac{(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}a_n|z|^{n-1}}$$

The last expression is less than $(1-\delta)$ if

$$\sum_{n=2}^{\infty}\frac{n-\delta}{1-\delta}a_n|z|^{n-1} < 1.$$

Using the fact that $f(z) \in TS_\lambda^m(\mu,\gamma)$ if and only if

$$\sum_{n=2}^{\infty}\frac{[2n-\mu(\gamma+1)]\phi_n(\lambda,m)}{(1-\gamma)}a_n \le 1.$$

We can say (2.11) is true if

$$\sum_{n=2}^{\infty}\frac{n-\delta}{1-\delta}|z|^{n-1} \le \frac{[2n-\mu(\gamma+1)\phi_n(\lambda,m)]}{(1-\gamma)}$$

or equivalently,

$$|z|^{n-1} \le \frac{(1-\delta)[2n-\mu(\gamma+1)]\phi_n(\lambda,m)}{(n-\delta)(1-\gamma)}$$

which yields the starlikeness of the family.

Integral Means Inequalities

In [6], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family *T*. He applied this function to resolve his integral means inequality conjuctured [13] and settled in [14], that

$\int_0^{2\pi} |f(re^{i\varphi})|^\eta d\varphi \leq \int_0^{2\pi} |f_2(re^{i\varphi})^\eta| d\varphi$

for all $f \in T, \eta > 0$ and $0 < r < 1$. In [12], he also proved his conjucture for the subclasses $T^*(\alpha)$ and $C(\alpha)$ of *T*.

Now, we prove Silverman's conjucture for the class of functions $TS_\lambda^m(\mu,\gamma)$. We need the concept of subordination between analytic functions and a subordination theorem of Littlewood [8].

Two functions *f* and *g*, which are analytic in *E*, the function *f* is said to be subordinate to *g* in *E* if exists a function *w* analytic in *E* with w(0)=0, $|w(z)| < 1$,($z \in E$) such that f(z)=g(w(z)), ($z \in E$).

We denote this subordination by *f(z)*p*g(z)*.(p denote subordination)

Lemma 1: If the function *f* and *g* are analytic in *E* with *f(z)*p *g(z),* then for $\eta > 0$ and $z = re^{i\varphi}$,$0 < r < 1$,

$$\int_0^{2\pi} |g(re^{i\varphi})|^\eta d\varphi \leq \int_0^{2\pi} |f(re^{i\varphi})|^\eta d\varphi$$

Now, we discuss the integral means inequalities for the functions *f* in $TS_\lambda^m(\mu,\gamma)$.

$$\int_0^{2\pi} |g(re^{i\varphi})|^\eta d\varphi \leq \int_0^{2\pi} |f(re^{i\varphi})|^\eta d\varphi$$

Theorem 7: Let $f \in TS_\lambda^m(\mu,\gamma)$,$0 \leq \mu \leq 1$,$0 \leq \gamma < 1$ and$f_2(z)$ be defined by

*f*2(*z*) = *z* − *φ*________2(*λ,m,μ,γ*1−*γ*)*z*2 (2.12)

Proof: For $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, (2.12) is equivalent to

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} a_n z^{n-1}\right|^\eta d\varphi \leq \int_0^{2\pi} \left|1 - \frac{1-\gamma}{\varphi_2(\lambda,m,\mu,\gamma)} z\right|^\eta d\varphi$$

By lemma 1, it is enough to prove that

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} p 1 - \frac{1-\gamma}{\varphi_2(\lambda,m,\mu,\gamma)} z$$

Assuming

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} p 1 - \frac{1-\gamma}{\varphi_2(\lambda,m,\mu,\gamma)} w(z)$$

and using (2.2) we obtain

$$|w(z)| = \left|\sum_{n=2}^{\infty} \frac{\varphi_2(\lambda,m,\mu,\gamma)}{1-\gamma} a_n z^{n-1}\right| \leq |z| \sum_{n=2}^{\infty} \frac{\varphi_2(\lambda,m,\mu,\gamma)}{1-\gamma} a_n \leq |z|$$

Where $\varphi_n(\lambda,m,\mu,\gamma) = [2n - \mu(\gamma + 1)]\phi_n(\lambda,m)$ This complete the proof.

CONCLUSION

This research paper has introduced a new subclass of univalent functions by means of Gegenbauer polynomials. For this subclass, some properties have been investigated; namely, coefficient bounds, growth and distortion theorems, radii of starlikeness, convexity and close-to-convexity are obtained. Furthermore, integral means inequality have been considered, inviting further research for this field of study.

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