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Common Fixed Point Theorem in M-Fuzzy Metric Space

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Abstract:

In this paper we prove a common fixed point result for six self mappings under weakly compatible condition in M-fuzzy metric space. This result generalizes and improves the results of many other authors existing in the literature.

MSC: primary 47H10; secondary 54H25.

Keywords: Fuzzy metric space; *M*-fuzzy metric space; weakly compatible self mappings; Fixed point

Introduction:

In 1965, the theory of fuzzy sets was investigated by Zadeh [10]. In the last many years there has been a great development and growth in fuzzy mathematics. Mustafa and Sims [7] firstly introduced G-metric space. Subsequently many authors have applied various form general topology of sets and developed the concept of fuzzy space. To use the concept of fuzzy topology and analysis in the theory of fuzzy sets and its applications have been developed by several eminent authors.

In 1975 Kramosil and Michalek [5] introduced the concept of fuzzy metric space which opened a new way for further development of analysis in such spaces. Several authors have introduced fuzzy metric space in different way. As George and Veeramani [2] modified the concept of a fuzzy metric space. Then we studied fuzzy metric space (shortly, FM space) then G-Fuzzy metric space (shortly, GF space). In 2006, Sedghi and Shobe [8] introduced D*-metric space as a probable modification of D-metric space and studied some topological properties which are not valid in D-metric spaces. Based on D∗- metric concepts, they [8] define M-fuzzy metric space and proved a common fixed point theorem for two mappings under the conditions of weak compatible and R-weakly commuting mappings in complete M-fuzzy metric spaces.

In this paper we prove a common fixed point result for six self mappings on a given set under weakly compatible condition in M-fuzzy metric space. Our result in this paper improve and generalize known result due to Saurabh Manro[6].

1. Preliminaries:

Definition 1.1 [10]

A subset *A* of universe set *X* with the membership function $\mu(x)$ which may take any value in the interval [0, 1] is called fuzzy set.

Definition 1.2 [8]

A binary operation $*:[0,1]\times[0,1]\rightarrow[0,1]$ is a continuous t-norm if it satisfies the following conditions:

(1) * is associative and commutative,

(2) * is continuous,

(3) $a * 1 = a$ for all $a \in [0,1]$,

(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

Example 1.3 [8]

Two examples of continuous t-norm are $a*b = ab$ and $a * b = min(a, b)$.

Definition1.4 [8]

The 3-tuple $(X, M, *)$ is known as fuzzy metric space (shortly, FM-space) if X is an any set, $*$ is a continuous t-norm, and M is a fuzzy set in $X \times X \times [0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t > 0$:

(FM-1) $M(x, y, 0) = 0$,

(FM-2) $M(x, y, t) = 1$ if and only if $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(FM-5) $M(x, y, \text{g})$: $[0, \infty) \to [0, 1]$ is left continuous.

Note that $M(x, y, t)$ can be thought of as a degree of nearness between x and y with respect to t.

Definition 1.7 [8]

A 3-tuple $(X, M, *)$ is called a M-fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm, and M is a fuzzy set on $X^3 \times (0, \infty)$, satisfying the following conditions for each $x, y, z, a \in X$ and $t, s > 0$,

(M1) $M(x, y, z, t) > 0$,

(M2) $M(x, y, z, t) = 1$ if and only if $x = y = z$,

(M3) $M(x, y, z, t) = M(p\{x, y, z\}, t)$, (symmetry) where p is a permutation function,

(M4) $M(x, y, a, t) * M(a, z, z, s) \leq M(x, y, z, t + s)$,

(M5) $M(x, y, z, \text{g}) : (0,\infty) \to [0, 1]$ is continuous.

Lemma 1.8 [8] If $(X, M, *)$ be a M-fuzzy metric space, then $M(x, y, z, t)$ is a non-decreasing with respect to t for all $x, y, z \in X$.

Proof: By taking $a = x$ and $z = x$ in the condition $M(x, y, a, t) * M(a, z, z, s) \leq M(x, y, z, t + s)$,

We get $M(x, x, y, t) \leq M(x, y, z, t + s)$,

If possible $M(x, y, z, t) > M(x, y, z, t + s)$,

Again if we put $a=x$ and $z=x$ in the condition $M(x, y, a, t) * M(a, z, z, s) \leq M(x, y, z, t + s)$,

We arrive a contradiction. Hence, the result.

2. Main Result:

Theorem 2.1 Let 3-tuple $(X, M, *)$ be a complete M-fuzzy metric space and let $\alpha, \beta, \gamma, \delta, \tau \text{ and } \upsilon$ be self mappings on X. Let the pairs $\{\alpha, \delta\}$ and $\{\beta, \tau\}$ and $\{\gamma, \upsilon\}$ be weak compatible. Also $\delta(X) \subset \beta(X),\ \tau(X) \subset \gamma(X),\ \upsilon(X) \subset \alpha(X)$. Also let $\alpha(X)$ is complete if there exist a $k \in (0,1)$ such that

$$M(\delta x, \tau y, \upsilon z, kt) \geq \max \begin{cases} M(\alpha x, \beta y, \gamma z, t) \\ M(\delta x, \alpha x, \beta z, t) \\ M(\tau x, \beta x, \beta z, t) \\ M(\upsilon x, \gamma x, \gamma z, t) \end{cases} ------(1)$$

For all $x, y, z \in X$. Then there exists a unique common fixed point of $\alpha, \beta, \gamma, \delta, \tau \text{ and } \upsilon$.

Proof: Since $\delta(X) \subset \beta(X),\ \tau(X) \subset \gamma(X),\ \upsilon(X) \subset \alpha(X)$. we can define sequences $\{x_m\}$ *and* $\{y_m\}$ *in* X. such that $y_{3m+1} = \delta x_{3m} = \beta x_{3m+1},\quad y_{3m+2} = \tau x_{3m+1} = \gamma x_{3m+2},\quad y_{3m+3} = \upsilon x_{3m+2} = \alpha x_{3m+3}$.

Then we have from equation (1);

$$M(\delta x_{3m},\ \tau x_{3m+1}, \upsilon x_{3m+2}, kt) \geq \max \begin{cases} M(\alpha x_{3m},\ \beta x_{3m+1},\ \gamma x_{3m+2}, t) \\ M(\delta x_{3m}, \alpha x_{3m}, \beta x_{3m+2}, t) \\ M(\tau x_{3m}, \beta x_{3m}, \beta x_{3m+2}, t) \\ M(\upsilon x_{3m}, \gamma x_{3m}, \gamma x_{3m+2}, t) \end{cases}$$

$$M(y_{3m+1}, y_{3m+2}, y_{3m+3}, kt) \geq \max \begin{cases} M(y_{3m}, y_{3m+1}, y_{3m+2},\ t) \\ M(y_{3m+1}, y_{3m}, y_{3m+2},\ t) \\ M(y_{3m+1}, y_{3m}, y_{3m+2},\ t) \\ M(y_{3m+1}, y_{3m}, y_{3m+2},\ t) \end{cases}$$

$$M(y_{3m+1}, y_{3m+2}, y_{3m+3}, kt) \geq M(y_{3m}, y_{3m+1}, y_{3m+2}, kt)$$

Similarly we have $M(y_m, y_{m+1}, y_{m+2}, kt) \geq M(y_{m-1}, y_m, y_{m+1}, t)$

Hence $\{y_m\}$ is a Cauchy and since X is complete, then there exists $z\ in\ X.$ such that $y_m \to z.$

So the subsequences $\{y_{3m}\}, \{y_{3m+1}\}, \{y_{3m+2}\}$ are also convergent.

That is $\lim \beta x_{3m+1} = \lim \delta x_{3m} = \lim \tau x_{3m+1} = \lim \alpha x_{3m+3} = \lim \upsilon x_{3m+2} = z.$

We claim that $\lim \delta w = z$

$$M(\delta w, \tau x_{3m+1}, \upsilon x_{3m+2}, kt) \geq \max \begin{cases} M(\alpha w, \beta x_{3m+1}, \gamma x_{3m+2}, t) \\ M(\delta w, \alpha w, \beta x_{3m+2}, t) \\ M(\tau w, \beta w, \beta x_{3m+2}, t) \\ M(\upsilon w, \gamma w, \gamma x_{3m+2}, t) \end{cases} \quad M(\delta w, y_{3m+2}, y_{3m+3}, kt) \geq \max \begin{cases} M(z, y_{3m+1}, y_{3m+2}, t) \\ M(\delta w, z, y_{3m+2}, t) \\ M(\delta w, z, y_{3m+2}, t) \\ M(\upsilon w, \gamma w, y_{3m+2}, t) \end{cases}$$

Taking limit $m \to \infty$

$$M(\delta w, z, z, kt) \geq \max \begin{cases} M(z, z, z, t) \\ M(\delta w, z, z, t) \\ M(\delta w, z, z, t) \\ M(\upsilon w, \gamma w, z, t) \end{cases}$$

That is $M(\delta w, z, z, kt) = 1.$

Therefore $\delta w = z = \alpha w.$

Hence w is the coincidence point of δ and α .

$\delta(X) \subset \beta(X), ie\, z \in \delta(X) \subset \beta(X),$

Then there must exists $t \in X$ s.t. $\beta t = z.$

$$M(\delta x_{3m},\tau t,\upsilon x_{3m+2},kt)\geq \max\begin{cases}M(\alpha x_{3m},\beta t,\gamma x_{3m+2},t)\\ M(\delta x_{3m},\alpha x_{3m},\beta x_{3m+2},t)\\ M(\tau x_{3m},\beta x_{3m},\beta x_{3m+2},t)\\ M(\upsilon x_{3m},\gamma x_{3m},\gamma x_{3m+2},t)\end{cases}$$

$$M(y_{3m+1},\tau t,y_{3m+2},kt)\geq \max\begin{cases}M(y_{3m},z,y_{3m+2},t)\\ M(y_{3m+1},y_{3m},y_{3m+2},t)\\ M(y_{3m+1},y_{3m},y_{3m+2},t)\\ M(y_{3m+1},y_{3m},y_{3m+2},t)\end{cases}$$

Taking limit $m\to\infty$

$$M(z,\tau t,z,kt)\geq \max\begin{cases}M(z,z,z,t)\\ M(z,z,z,t)\\ M(z,z,z,t)\\ M(z,z,z,t)\end{cases}$$

That is $M(z,\tau t,z,kt)=1$

Then $\tau t=z=\beta t,$

Thus *t* is a coincidence point if $\beta\, and\, \tau.$

Now $\tau(X)\subset\gamma(X),\quad i.e.\quad z=\tau t\in\tau(X)\subset\gamma(X).$

Then there exists $v\in X$ such that $\gamma v=z.$

$$M(\delta x_{3m},\tau x_{3m+1},\upsilon v,kt)\geq \max\begin{cases}M(\alpha x_{3m},\beta x_{3m+1},\gamma v,t)\\ M(\delta x_{3m},\alpha x_{3m},\beta v,t)\\ M(\tau x_{3m},\beta x_{3m},\beta v,t)\\ M(\upsilon x_{3m},\gamma x_{3m},\gamma v,t)\end{cases}$$

$$M(y_{3m+1},y_{3m+2},\upsilon v,kt)\geq \max\begin{cases}M(y_{3m},y_{3m+1},\gamma v,t)\\ M(y_{3m+1},y_{3m},\beta v,t)\\ M(y_{3m+1},y_{3m},\beta v,t)\\ M(y_{3m+1},y_{3m},\gamma v,t)\end{cases}$$

Taking limit $m\to\infty$

$M(z,z,\upsilon v,kt)\geq 1.$

Hence $\upsilon v=z.$ we have $\upsilon v=\gamma v=z.$

Since $\{\alpha,\delta\}$ and $\{\beta,\tau\}$ and $\{\gamma,\upsilon\}$ be weak compatible, they commute at coincidence points.

We have $\delta w=z=\alpha w.$ then $\alpha\delta w=\delta\alpha w\quad i.e.\quad \alpha z=\delta z.$ also $\tau t=z=\beta t.\quad then\quad \beta\tau t=\tau\beta t$ that is $\beta z=\tau z.$ since $\{\gamma,\upsilon\}$ is weakly compatible, similarly we get $\gamma z=\upsilon z$

$$M(\delta z, \tau x_{3m+1}, \upsilon z, kt) \geq \max \begin{cases} M(\alpha z, \beta x_{3m+1}, \gamma z, t) \\ M(\delta z, \alpha z, \beta z, t) \\ M(\tau z, \beta z, \beta z, t) \\ M(\upsilon z, \gamma z, \gamma z, t) \end{cases}$$

Taking limit $m \to \infty$

$$M(\delta z, z, \upsilon z, kt) \geq \max \begin{cases} M(\alpha z, z, \gamma z, t) \\ M(\delta z, \alpha z, \beta z, t) \\ 1 \\ 1 \end{cases}$$

Thus we have $\delta z = z = \upsilon z$. Hence $\alpha z = \delta z = \gamma z = \upsilon z = z$.

$$M(\delta x_{3m}, \tau z, \upsilon z, kt) \geq \max \begin{cases} M(\alpha x_{3m}, \beta z, \gamma z, t) \\ M(\delta x_{3m}, \alpha x_{3m}, \beta v, t) \\ M(\tau x_{3m}, \beta x_{3m}, \beta v, t) \\ M(\upsilon x_{3m}, \gamma x_{3m}, \gamma v, t) \end{cases}$$

Taking limit $m \to \infty$

$$M(z, \tau z, \upsilon v, kt) \geq \max \begin{cases} M(z, \beta z, \gamma z, t) \\ M(z, z, \beta z, t) \\ M(z, z, \beta v, t) \\ M(z, z, z, t) \end{cases} \geq \max \begin{cases} M(z, \beta z, \gamma z, t) \\ M(z, z, \beta z, t) \\ M(z, z, \beta v, t) \\ 1 \end{cases}$$

Hence $\tau z = \upsilon z = z.$

Thus $\alpha z = \delta z = \beta z = Tz = \upsilon z = \gamma z = z.$

Thus *z* is a common fixed point of the self mappings $\alpha, \beta, \gamma, \delta, \tau$ *and* υ.

To prove uniqueness of fixed point, let *y* be another fixed point of the self mappings $\alpha, \beta, \gamma, \delta, \tau$ *and* υ.

$$M(\delta z, \tau y, \upsilon z, kt) \geq \max \begin{cases} M(\alpha z, \beta y, \gamma z, t) \\ M(\delta z, \alpha z, \beta z, t) \\ M(\tau z, \beta z, \beta z, t) \\ M(\upsilon z, \gamma z, \gamma z, t) \end{cases}$$

$$M(z, y, z, kt) \geq \max \begin{cases} M(z, y, z, t) \\ M(z, z, z, t) \\ M(z, z, z, t) \\ M(z, z, z, t) \end{cases}$$

$$M(z, y, z, kt) \geq 1.$$

Hence $y = z$.

3. Conclusion:

Fixed point theory has many applications in several branches of science such as game theory, nonlinear programming, economics, theory of differential equations, etc. in this paper we prove common fixed point theorem in M-fuzzy metric space. Our result presented in this paper generalized and improve some known result in fuzzy metric space.

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