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An Unified Approach for the Contracting Mapping in Fixed Point Theorem

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Abstract:

This paper establishes and proves a fixed point theorem for [15-17] type contraction in complete metric space.

In doing so, we have extend a result existing [16] into complete metric space. An illustrative example is given to demonstrate the validity of our results.

Keywords: Non-negative numbers, contraction mapping, complete metric space, etc.

Introduction:

The well-known Banach fixed point theorem is often referred to as Banach contraction principle which appeared in its explicit form in the thesis of Banach in 1922 (see [1]). Fixed point theory is a rich, interesting and exciting branch of Mathematics. It is relatively young but fully developed area of research. The study of existence of fixed points falls within the several domains such as: classical analysis, functional analysis, operator theory, topology, and algebraic topology. Fixed point theorems have found wide applications in the theory of non-linear oscillations, fluid flow, theory of games, approximation theory, and initial and boundary value problems for ordinary and partial differential equations (see [2, 3]). This natural theorem asserts that every contraction mapping defined on a complete metric space has a unique fixed point and that fixed point can be explicitly obtained as limit of repeated iteration of the mapping of any point of the underlying space. Every contraction mapping is a continuous but not conversely. A result on the existence and uniqueness of common fixed point in metric spaces generally involves conditions on commutativity, continuity, and contraction along with a suitable condition on the containment of range of one mapping into the range of the other. Thus, one is always required to improve one or more of these conditions to prove a new fixed point theorem. In recent past, several authors have contributed to the vigorous development of metric fixed point theory see [4, 11]. In metric fixed point theory, the first ever attempt to coin an implicit relation can be traced back to Popa [12]. In 2008, Ali and Imdad (see [13]) introduced a new class of implicit functions which covers several classes of contraction conditions such as Ćirić quasi-contraction, generalized contractions, ψ -type contractions, rational inequalities, and among others and they proved some fixed point theorems for weakly compatible mapping satisfying common property.

1. Definition: (see [13]): Let (M, P) be a complete metric space and ϕ is a mapping of M into itself which satisfies the following condition:

$P(\phi a,\phi b)\le c\,\rho(a,b)$ for some $c<1$ and $a,b\in M$, then ϕ has a unique fixed point ξ for $a\in M$.

2. Definition (see [14]): We investigate mappings which satisfy the following condition:

$$\rho(\phi a,\phi b)\le \psi\rho(a,b)$$

where ψ is some function defined on the closure of the range of ρ.

3. Definition (see [13]): A self mapping of a metric space (M, d) is called a ρ-contraction if there exist a map $q\in P$ (P is a set) such that, for all $a,b\in M$

$$\rho(\phi a,\phi b)\le q\big(\rho(a,b),\rho(a,\phi a),\rho(b,\phi b)\big).$$

4. Definition (see [13], [15]): There exists three non-negative numbers x, y, z such that $\rho(\phi a,\phi b)\le x\,\rho(a,b)+y\,\rho(a,\phi a)+z\,\rho(b,\phi b)$ where $x+y+z<1$.

5. Definition (see [15]): There exists three non-negative numbers x,y,z such that

$$x(w)+y(w)+z(w)<1.$$

<u>Known theorems:</u>

In 1971, S. Reich prove that following theorem:

Let (M, P) be a complete metric space and $\phi_k\ (k=1,2,3,\ldots)$ be a mappings of M into itself which satisfy the above definition with the fixed points g_ℓ. Suppose that a mapping ϕ of M into itself can be defined by

$\phi a=\lim_{\ell\to\infty}\phi_\ell a$ then

$g=\lim_{\ell\to\infty}g_\ell$ is the unique fixed point of ϕ.

<u>Main theorem:</u>

Let (M, P) be a complete metric space and $\phi_\ell\ (\ell\ge 1)$ be a mappings of M into itself which satisfy

$$x(w)+y(w)+z(w)<1,$$

$$\lim_{\zeta\to w+}\sup\big[x(\zeta)+y(\zeta)+z(\zeta)\big]<1$$

and

$\rho(\phi a,\phi b)\le x\big(\rho(a,b)\big).\rho(a,\phi a)+y\big(\rho(a,b)\big).\rho(b,\phi b)+z\big(\rho(a,b)\big).\rho(a,b)$ with fixed points g_ℓ.

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Suppose that a self-mapping of M can be defined by $\phi a = \lim_{\ell \to \infty} \phi_\ell a$ then $g = \lim_{\ell \to \infty} g_\ell$ is the unique fixed point.

For the proof of the above, we need the following example.

Example: Let Z^+ denote the set of positive integers.

Let $M = [0,1] \cup \{z^+ - \{1\}\}$ equipped with metric ρ defined as follows:

$$\rho(a,b) = \begin{cases} |a-b|, & a,b \in [0,1] \\ a+b, & a,b \in R - \{1\} \end{cases}$$

then (M, d) is a complete metric space because it is isometric to a closed subspace of the space of absolutely summable sequences.

Now, we define $\phi_\ell : M \to M\left(\ell \in z^+\right)$, by setting $\phi_1(a) = \phi_2(a) = 0, a \in M$, for $\ell \geq 3$

$$\phi_\ell(a) = \begin{cases} a-1, & a \in z^+ - \{1\} \\ a - \dfrac{\ell-1}{2\ell-3}, a^2, a \in [0,1] \end{cases}.$$

Then $a_\ell = 0$ is only fixed point of ϕ_ℓ. For $\ell \geq 3, \phi_\ell$ doesnot satisfy the above definition.

Otherwise for $b = 0$ and $a \in (0,1]$,

We have

$$a - \frac{\ell-1}{2\ell-1}.a^2 \leq \frac{1}{2}\frac{\ell-1}{2\ell-3}.\ell^2 + y(b).b.$$

Without loss of generality, we put $y < \frac{1}{2}$.

For any $a \in (0,1]$,

$$1 - \frac{\ell-1}{2\ell-3}a < \frac{1}{2}\frac{\ell-1}{2\ell-3}.a + z(a).$$

Now for $\ell \geq 3$,

$$1 - a < 1 - \frac{1}{2}\frac{\ell-1}{2\ell-3}.a < z(a)$$

thus for $a \in (0,1]$, $\ell \geq 3$

$$x(a) + y(a) < 1 - z(a) < a.$$

Also, for $a \in (0,1]$,

$$x(a) = y(a) = 0.$$

Thus from given definition, $\rho(\phi_\ell a,\phi_\ell b) \le k(\rho(a,b).\rho(a,b))$

for $a=0$ and $b \in z^{+}-\{1\}$,

$w-1 \le z(w).w$

$=) \frac{w-1}{w} \le z(1)$.

$=) \frac{w-1}{w} \le z(1)$.

Thus we obtain $1 \le z(1) < 1$

which gives a contradiction for $\ell \in z^{+}$,

$x(w)=y(w)=0, w>0$

and

$$z(w)=\begin{cases} 1-\frac{1}{2}w, & w\in[0,1] \\ 1-\frac{1}{w}, & w>1 \end{cases}.$$

For $\ell \ge 3,\ a,b\in[0,1], a\ne b, \rho(\phi_\ell a,\phi_\ell b)=(a-b)\left[1-\frac{\ell-1}{2\ell-1}, a+b\right]$,

$\le (b-a).\left[1-\frac{1}{a+b}\right]$

$=k\rho(a,b).\rho(a,b)$,

For $a,b \in z^{+}-\{1\}, \ell \ge 3$,

$\rho(\phi_\ell a,\phi_\ell b)=\phi_\ell a+\phi_\ell b=(a+b-1)$

$=(a+b)\left[1-\frac{1}{a+b}\right]$

$=k(\rho(x,y).\rho(a,b))$.

We define

$$\phi(a)=\begin{cases} a-\frac{a^2}{2}, & a\in[0,1] \\ a=1, & a\in z^{+}-\{1\} \end{cases}.$$

Then $g=0$ is the only fixed point of ϕ for all $a\in M$.

$=) \lim_{x \to \infty} \phi_\ell(a) = \phi(a)$.

Clearly the sequence $\{g_\ell\}$ converges to g.

Thus the conclusion of the theorem holds good.

Conclusion:

Banach's fixed point theorem has been extensively studies to solve the problems in non-linear analysis since many years. This theorem provided the existence and uniqueness of the solution. It states that if (M, ρ) is a complete metric space and $\phi : M \to M$ is self contractive mapping. Then ϕ has unique fixed point $g \in M$. 1969, Boyd and Wong [17] gave an important generalization of the Banach fixed point theorem by the application of control function in the Banach contraction condition. They took into account the condition at in definition (2) where (M, ρ) is a complete metric space, and a mapping $\phi : [0, \infty) \to [0, \infty)$ is upper semi continuous from the right on $[0, \infty)$ such that $\zeta(w) < w$.

$\therefore$ ϕ is a unique fixed point $g \in M$. In the same way, the main contribution to the fixed point theorem is our definition, theorem and example. here, the results have proved for the result [16] type contraction in complete metric spaces. Several existing results in the literature are generalized and extended into complete metric space. Suitable examples are given to demonstrate the validity of the results.

Finally, the existence of the solution of non-linear integral equation is discussed as the application of the main result.

References:

[1] S. Banach, Sur less operations dans les ensembles obstraits et leur application aux equations integrals, Fundamenta Mathematicae, Vol. 3, pp. 131-181, 1922.

[2] W.A. Kirk, Some recents results in metric fixed point theory, Journal of fixed point theory and applications, Vol. 2, no. 2, pp. 195-207, 2007.

[3] C. Semple and M. Steel, Phylogenetics, Vol. 24 of Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, Oxford, UK, 2003.

[4] M. Imdad, M.S. Khan, and S. Sessa, One some weak conditions of commutativity in common fixed point theorem, International Journal of Mathematics and Mathematical Sciences, Vol. 11, no. 2, pp. 289-296, 1988.

[5] L. Ciric, A. Razani, S. Radenovie, and J. S. Ume, Common fixed point theorems for families of weakly compatible maps, computers and Mathematics with Applications, Vol. 55, no. 11, pp. 2533-2543, 2008.

[6] M. Imdad, S. Chauhan, and Z. Kadelburg, Fixed point theorem for mappings with common limit range property satisfying generalized (ψ, ϕ)-weak contractive conditions, Mathematical Sciences, Vol. 7, article 16, 2013.

[7] M. Imdad and J. Ali, Jungck's common fixed point theorem and E. A. property, Acta Mathematica Sinica, Vol. 24, no. 1, pp. 87-94, 2008.

[8] L.B. Ciric, Generalized contractions and fixed point theorem, publications de liinstitut Mathematique, Vol. 12, no. 26, pp. 19-26, 1971.

[9] M. Imdad and Q. H. Khan, Six mappings satisfying a rational inequality, Radio Mathematicki, Vol. 9, no. 2, pp. 251-260, 1999.

[10] W. Shatanawi, Fixed point theorems for nonlinear weakly C-Contractive mapping in metric spaces, Mathematical and Computer Modeling, Vol. 54, no. 11-12, pp. 2816-2826, 2011.

[11] B.E. Rhoades, Some theorems on weakly contractive maps, in Proceedings of the 3rd World Congress of Nonlinear Analysis, Vol. 47, pp. 2683-2693, 2001.

[12] V. Popa, Some fixed point theorems for weakly compatible mappings, Radio, Mathematicki, Vol. 10, no. 2, pp. 245-252, 2001.

[13] D. Delbosco, An Unified Approach for the contractive Mappings, Jnanabha, Vol. 16, 1986, 1-11.

[14] E. Rakotch, A note on contractive mappings, proc. Amer. Math. Soc. 13, 1962, 459-465.

[15] S. Reich, Some remarks concerning contraction mappings, Canad. Math. Bull. 14, (1971), 121-124.

[16] S. Reich, Kannan's fixed point theorem, Bull, Un. Mat. Ital. 4, 1971, 1-11.

[17] D.W. Boyd and J.S. Wong: On non-linear contractions, Proc. Amer. Math. Soc., Vol. 20, no. 2, 1969, 458-464.