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A Modified Ostrowski Inequality with Random Variable Application on $L_p[a,b], 0 < p < 1$,Spaces

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Abstract:

Many outhers proved types of Ostrowski inequality. We improve their inequalities and then applied it to cumulative density function and **Beta** and normal distribution:

1.1. Introduction

In 1938 Ostrowski introduced his famous inequality in [8]

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \left[\frac{1}{4}+\left(\frac{x-{(a+b)}/{2}}{b-a}\right)^2\right](b-a)M \qquad (1)$$

For a differentiable function f with bounded derivative on (a,b).

In [4] Drangomir and Wang proved the of Ostrowski's inequality

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt-\frac{f(b)-f(a)}{b-a}\right| \le \frac{1}{4}(b-a)(C-D) \qquad (2)$$

For a function with bounded derivative, such that $C \le \acute{f} \le D$ on (a,b). C, D are positive constants in R.

In [4] Matid, Decarce and Ujevic proved (2) with $\frac{1}{4\sqrt{3}}$ constant with twice diffentiable formula on (a,b).Using Chebychev's operator we improve the results in[6], by introducing a best result for functions in $L_p[a,b]$ for $0 < p < 1$. And we assume that $\acute{f}$ is also in $L_p[a,b]$. Then we applied our inequality to $Beta$ and Normal distribution and cumulative density function.

1.2. The Main Result

To prove our main theorem we need the following auxiliary result.

Lemma 1.2.1[3]

$$f(x)-\frac{1}{b-a}\int_a^b f(t)dt-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)$$
$$=\frac{1}{2(b-a)^2}\int_a^b\int_a^b (P(x,t)-P(x,s))(\acute{f}(t)-\acute{f}(s))dtds,$$

where

$$P(X,t)=\begin{cases} t-a & if \quad t\in[a,X) \\ t-b & if \quad t\in(X,b]\end{cases}$$

Theorem 1.2.2

Let X be a random variable $f, f' \in L_p[a,b]$, $0 < p < 1$, then,

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right|$$

$$\leq \left(\frac{c(p)(b-a)^{p-1}}{2(p+1)}\right)\left[\|\acute{f}\|_p + \left(\left(\frac{f(b)-f(a)}{b-a}\right)^p\right)^{\frac{1}{p}}\right],$$

Proof:

By using the Gaushy-Schwars inequality for double integrals, we get,

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b (P(x,t)-P(x,s))\left(\acute{f}(t)-\acute{f}(s)\right)dtds$$

$$\leq \left(\frac{1}{2(b-a)^2}\int_a^b\int_a^b \left(P(x,t)-P(x,s)\right)dtds\right)^{\frac{1}{2}} \circ$$

$$\left(\int_a^b\int_a^b \left(\acute{f}(t)-\acute{f}(s)\right)dtds\right)^{\frac{1}{2}},$$

Since any two norms on a finite dimensional space are equivalent, then we get,

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b (P(x,t)-P(x,s))\left(\acute{f}(t)-\acute{f}(s)\right)dtds$$

$$\leq \left(\frac{c}{2(b-a)^2}\int_a^b\int_a^b |P(x,t)-P(x,s)|^p\, dtds\right)^{\frac{1}{p}} \circ$$

$$\left(\frac{1}{2(b-a)^2}\int_a^b\int_a^b \left|\acute{f}(t)-\acute{f}(s)\right|^2 dtds\right)^{\frac{1}{2}}, \qquad (3)$$

where c is a positive constant.

Now, using (3), we get

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b (P(x,t)-P(x,s))\left(\acute{f}(t)-\acute{f}(s)\right)dtds$$

$$\leq \left(\frac{c(p)}{2(b-a)^2}\int_a^b\int_a^b |P(x,t)-P(x,s)|^p\, dtds\right)^{\frac{1}{p}} \circ$$

$$\left(\frac{c(p)}{2(b-a)^2}\int_a^b\int_a^b \left|\acute{f}(t)-\acute{f}(s)\right|^p dtds\right)^{\frac{1}{p}}$$

$$= \mathrm{I}_1 \circ \mathrm{I}_2 \qquad (4)$$

Now,

$$I_1 = \left(\frac{cc(p)}{2(b-a)^2}\int_a^b\int_a^b |P(x,t)-P(x,s)|^p\, dtds\right)^{\frac{1}{p}} \leq \left[\frac{c(p)}{2(b-a)^2}\int_a^b\int_a^b |p(x,t)|^P + |p(x,s)|^P\, dtds\right]^{\frac{1}{p}}.$$

This implies,

$$I_1 = \left(\frac{c(p)}{2(b-a)^2} \int_a^b \int_a^b |P(x,t) - P(x,s)|^p \, dtds \right)^{\frac{1}{p}}$$

$$\leq \left[\frac{c(p)}{2(b-a)^2} \left(\int_a^b \int_a^b |p(x,t)|^p \, dtds + \int_a^b \int_a^b |p(x,s)|^p \, dtds \right) \right]^{\frac{1}{p}}.$$

Then,

$$I_1 = \left(\frac{c(p)}{2(b-a)^2} \int_a^b \int_a^b |P(x,t) - P(x,s)|^p \, dtds \right)^{\frac{1}{p}}$$

$$\leq \left[\frac{c(p)}{2(b-a)^2} \left((b-a) \int_a^b |p(x,t)|^p \, dt + (b-a) \int_a^b |p(x,s)|^p \, ds \right) \right]^{\frac{1}{p}}.$$

This implies,

$$I_1 = \left(\frac{c(p)}{2(b-a)^2} \int_a^b \int_a^b |P(x,t) - P(x,s)|^p \, dtds \right)^{\frac{1}{p}}$$

$$\leq \left[\frac{c(p)}{2(b-a)} \left(\int_a^b |p(x,t)|^p \, dt + \int_a^b |p(x,s)|^p \, ds \right) \right]^{\frac{1}{p}}.$$

By using assumption in Lemma 1.2.1, we get,

$$I_1 = \left(\frac{c(p)}{2(b-a)^2} \int_a^b \int_a^b |P(x,t) - P(x,s)|^p \, dtds \right)^{\frac{1}{p}}$$

$$\leq \left[\frac{c(p)}{2(b-a)} \left[\left(\int_a^x (t-a)^p \, dt + \int_x^b (t-b)^p \, dt \right) + \left(\int_a^x (s-a)^p \, ds + \int_x^b (s-b)^p \, ds \right) \right] \right]^{\frac{1}{p}}$$

Therefore,

$$I_1 = \left(\frac{c(p)}{2(b-a)^2} \int_a^b \int_a^b |P(x,t) - P(x,s)|^p \, dtds \right)^{\frac{1}{p}}$$

$$\leq \left[\frac{c(p)}{2(b-a)} \left(\frac{1}{p+1} (t-a) \right)^{p+1} \Bigg|_a^x + \left(\frac{1}{p+1} (t-b) \right)^{p+1} \Bigg|_x^b \right]^{\frac{1}{p}}.$$

Then,

$$I_1 = \left(\frac{c(p)}{2(b-a)^2} \int_a^b \int_a^b |P(x,t) - P(x,s)|^p \, dtds \right)^{\frac{1}{p}}$$

$$\leq c(p) \left[\frac{c(p)}{2(b-a)(p+1)} \left((x-a) \right)^{p+1} - \left((x-b) \right)^{p+1} \right]^{\frac{1}{p}}$$

$$\leq \left(\frac{2c(p)}{2(b-a)(p+1)} (b-a)^{p+1} \right)^{\frac{1}{p}}.$$

This implies,

$$I_1 = \left(\frac{c(p)}{2(b-a)^2} \int_a^b \int_a^b |P(x,t) - P(x,s)|^p \, dtds \right)^{\frac{1}{p}} \leq \left(\frac{c(p)(b-a)^p}{(p+1)} \right)^{\frac{1}{p}}. \quad (5)$$

Also,

$$I_2 = \left(\frac{c(p)}{2(b-a)^2} \int_a^b \int_a^b \left| f'(t) - f'(s) \right|^p dtds \right)^{\frac{1}{p}}$$

$$\leq \left[\frac{c(p)}{2(b-a)^2} \left(\iint_a^b \left| f'(t) \right|^p dtds + \iint_a^b \left| f'(s) \right|^p dtds \right) \right]^{\frac{1}{p}},$$

Since $\|\circ\|_p < \|\circ\|_1$, we get,

$$I_2 = \left(\frac{c(p)}{2(b-a)^2} \int_a^b \int_a^b \left| f'(t) - f'(s) \right|^p dtds \right)^{\frac{1}{p}}$$

$$\leq \left[c(p) \frac{c}{2(b-a)^2} \left((b-a) \int_a^b \left| f'(t) \right|^p dt + (b-a) \int_a^b \left| f'(s) \right|^p ds \right) \right]^{\frac{1}{p}}$$

$$\leq \frac{c(p)}{2(b-a)^2} (b-a) \left(\left\| f' \right\|_p + \int_a^b \left| f'(s) \right|^p ds \right)$$

Therefore,

$$I_2 = \left(\frac{c(p)}{2(b-a)^2} \int_a^b \int_a^b \left| f'(t) - f'(s) \right|^p dtds \right)^{\frac{1}{p}} \leq \left(\frac{c(p)}{2(b-a)} \right)^{\frac{1}{p}} \left(\left\| f' \right\|_p + \frac{f(b) - f(a)}{b-a} \right). \quad (6)$$

Put (5) and (6) in (4), we get,

$$\left| \frac{1}{2(b-a)^2} \int_a^b \int_a^b (P(x,t) - P(x,s)) \left(f'(t) - f'(s) \right) dtds \right|$$

$$\leq \left(\frac{c(p)(b-a)^p}{(p+1)} \right)^{\frac{1}{p}} \left(\frac{c(p)}{2(b-a)} \right)^{\frac{1}{p}} \left(\left\| f' \right\|_p + \frac{f(b) - f(a)}{b-a} \right). \quad (7)$$

By using Lemma 1.2.1 and (7), we get,

$$\left| \boldsymbol{f(x) - \frac{1}{b-a} \int_a^b f(t) dt - \frac{f(b) - f(a)}{b-a} \left(x - \frac{a+b}{2} \right)} \right|$$

$$\leq \left(\frac{c(p)(b-a)^{p-1}}{2(p+1)} \right)^{\frac{1}{p}} \left[\left\| f' \right\|_p + \left(\frac{f(b) - f(a)}{b-a} \right)^p \right]$$

where $\boldsymbol{c(p)}$ is a positive constant depending on p only ■

Collorary 6.2.3

Let $\boldsymbol{X}$ be a random variable $\boldsymbol{f, f'} \in \mathbf{L_p[a,b]}$, $\mathbf{0 < p < 1}$, then

$$\left| \boldsymbol{f\left(\frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t) dt} \right| \leq \left(\boldsymbol{\frac{c(p)(b-a)^{p-1}}{2(p+1)}} \right)^{\boldsymbol{\frac{1}{p}}} \left[\left\| \boldsymbol{f'} \right\|_{\boldsymbol{p}} + \left(\boldsymbol{\frac{f(b) - f(a)}{b-a}} \right)^{\boldsymbol{p}} \right].$$

Proof:

Take $X = \frac{a+b}{2}$ is the midpoint,

By using Theorem 1.2.2, we get,

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left(\frac{c(p)(b-a)^{p-1}}{2(p+1)}\right)^{\frac{1}{P}}\left[\|\acute{f}\|_p+\left(\frac{f(b)-f(a)}{b-a}\right)^p\right],$$

where *c(p)* is a positive constant depending upon p.

1.3.Applications for Distribution Function and Random Variable.

Theorem 1.3.1

Let X be a random variable with probability density function
$f:[a,b] \to R_+$ and having the cumulative density function
$F:[a,b] \to [0,1]$ and $f, F \in L_p[a,b]$, $0 < p < 1$, then,

$$\left|F(x)-\frac{b-E(X)}{b-a}-\frac{1}{b-a}\left(x-\frac{a+b}{2}\right)\right| \leq \left(\frac{c(p)(b-a)^{p-1}}{2(p+1)}\right)^{\frac{1}{P}}\left[\|f\|_p+\left(\frac{1}{b-a}\right)^p\right],$$

where $\boldsymbol{c(p)}$ is a positive constant depending on p only .

Proof:

Put F(x) instead of f in Theorem 1.2.2, we get,

$$\left|F(x)-\frac{1}{b-a}\int_a^b F(t)dt-\frac{F(b)-F(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right|$$

$$\leq \left(\frac{c(p)(b-a)^{p-1}}{2(p+1)}\right)^{\frac{1}{P}}\left[\|\acute{F}\|_p+\left(\frac{F(b)-F(a)}{b-a}\right)^p\right] \qquad (8)$$

Since $F(a) = 0$ and $F(b) = 1$. (9)

Also, since $F(x) = \int_a^X f(t)dt,\ X \in [a,b]$.

So, $\acute{F} = f$ and E(X) $\int_a^b tdF(t)dt$.

$$\int_a^b F(t)dt = b - E(X). \qquad (10)$$

Put (9) and (10) in (8), we get,

$$\left|F(x)-\frac{b-E(X)}{b-a}-\frac{1}{b-a}\left(x-\frac{a+b}{2}\right)\right|$$

$$\leq \left(\frac{c(p)(b-a)^{p-1}}{2(p+1)}\right)^{\frac{1}{P}}\left[\|f\|_p+\left(\frac{1}{b-a}\right)^p\right] \qquad \blacksquare$$

Colloraly 1.3.2

Let X be a random variable with probability density function
f and having the cumulative density function F ,such that
$f, F \in L_p[a,b]$, $0 < p < 1$, then,

$$\left|P_r\left(X \leq \frac{a+b}{2}\right)-\frac{b-E(X)}{b-a}\right|$$

$$\leq \left(\frac{c(p)(b-a)^{p-1}}{2(p+1)}\right)^{\frac{1}{P}}\left[\|f\|_p+\left(\frac{1}{b-a}\right)^p\right]$$

Proof:

Take $X = \frac{a+b}{2}$. Since $P_r(X \le x) = F(X)$, then by using Theorem 1.3.1, we get

$$\left|\boldsymbol{P_r}\left(\mathbf{X} \le \frac{\mathbf{a+b}}{\mathbf{2}}\right) - \frac{\mathbf{b - E(X)}}{\mathbf{b-a}}\right| \le \left(\frac{\boldsymbol{c}(\mathbf{p})(\boldsymbol{b-a})^{\boldsymbol{p}-\mathbf{1}}}{\mathbf{2}(\boldsymbol{p}+\mathbf{1})}\right)^{\frac{\mathbf{1}}{\boldsymbol{p}}}\left[\|\boldsymbol{f}\|_{\boldsymbol{p}} + \left(\frac{\mathbf{1}}{\boldsymbol{b-a}}\right)^{\boldsymbol{p}}\right] \qquad \blacksquare$$

The following theorem related to application for βeta distribution;

Theorem 1.3.3

Let X be a random variable and $X \sim B_{\alpha,\beta}$ whose the density function belongs to the L_p where $0 < p < 1$ then

$$\left|\boldsymbol{P_r}(\mathbf{X} \le \mathbf{x}) + \frac{\boldsymbol{\beta}}{\boldsymbol{\alpha}+\boldsymbol{\beta}} - \boldsymbol{X} + \frac{\mathbf{1}}{\mathbf{2}}\right|$$

$$\le \left(\frac{\boldsymbol{c}(\boldsymbol{p})}{\mathbf{2}(\boldsymbol{p}+\mathbf{1})}\right)^{\frac{\mathbf{1}}{\boldsymbol{p}}}\left[\frac{\mathbf{1}}{\boldsymbol{B}(\boldsymbol{\alpha},\boldsymbol{\beta})}(\boldsymbol{B}(\boldsymbol{P}(\boldsymbol{\alpha}-\mathbf{1})+,\boldsymbol{P}(\boldsymbol{\beta}-\mathbf{1})+\mathbf{1})^{\frac{\mathbf{1}}{\boldsymbol{p}}} + \mathbf{1}\right]$$

Proof:

Since $X \sim B_{\alpha,\beta}$, then the probability density function as follows,

$$f(X,\alpha,\beta) = \frac{X^{\alpha-1}(1-X)^{\beta-1}}{\beta(\alpha,\beta)} \qquad 0 < X < 1,$$

where, $B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt$.

Then

$$\|f(.,\alpha,\beta)\|_p = \frac{1}{B(\alpha,\beta)}(B(P(\alpha-1)+,P(\beta-1)+1)^{\frac{1}{p}}.$$

Also, $E(X) = \frac{\alpha}{\beta+\alpha}$.

Now, by using Theorem 1.3.1, we get,

$$\left|F(x) - \frac{b - \frac{\alpha}{\beta+\alpha}}{b-a} - \frac{1}{b-a}\left(x - \frac{a+b}{2}\right)\right|$$

$$\le \left(\frac{c(p)(b-a)^{p-1}}{2(p+1)}\right)^{\frac{1}{p}}\left[\frac{1}{B(\alpha,\beta)}(B(P(\alpha-1)+,P(\beta-1)+1)^{\frac{1}{p}} + \left(\frac{1}{b-a}\right)^{p}\right].$$

Since $P_r(X \le x) = F(X)$, and b=1, then

$$\left|\boldsymbol{P_r}(\mathbf{X} \le \boldsymbol{x}) + \frac{\boldsymbol{\beta}}{\boldsymbol{\alpha}+\boldsymbol{\beta}} - \boldsymbol{x} + \frac{\mathbf{1}}{\mathbf{2}}\right|$$

$$\le \left(\frac{\boldsymbol{c}(\boldsymbol{p})}{\mathbf{2}(\boldsymbol{p}+\mathbf{1})}\right)^{\frac{\mathbf{1}}{\boldsymbol{p}}}\left[\frac{\mathbf{1}}{\boldsymbol{B}(\boldsymbol{\alpha},\boldsymbol{\beta})}(\boldsymbol{B}(\boldsymbol{P}(\boldsymbol{\alpha}-\mathbf{1})+,\boldsymbol{P}(\boldsymbol{\beta}-\mathbf{1})+\mathbf{1})^{\frac{\mathbf{1}}{\boldsymbol{p}}} + \mathbf{1}\right] \qquad \blacksquare$$

The next theorem is an application for Normal distribution.

Theorem 1.3.4

Let X be a random variable with parameters $(\mu,\delta^2) \in \Omega$ **,** with the probability density function

$f(X,\mu,\delta^2) = \frac{1}{\delta\sqrt{2\pi}} e^{\frac{-(X-\mu)^2}{2\delta^2}}$ **,**

where $\Omega = \{(\mu,\delta^2); -\infty < \mu < \infty, 0 < \delta^2 < \infty\}$.

Then

$$\left|P_r\left(X \le \left(\frac{a+b}{2}\right)\right) + \mu - X - \frac{1}{2}\right| \le \left(\frac{c(P)}{2(p+1)}\right)^{\frac{1}{P}}\left[\left(\frac{2}{\pi}\right)^{\frac{1}{2p}}\left(e^{\frac{\mu}{\delta}}\right)^{\frac{1}{p}} + 1\right]$$

Proof:

Since $X \sim N(\mu, \delta^2)$,in L_p where $0 < p < 1$ then we get,

$\|f\|_p \leq \left(\frac{2}{\pi}\right)^{\frac{1}{2p}} \left(e^{\frac{\mu}{\delta}}\right)^{\frac{1}{p}}$.Also, $E(X) = \mu$, $F(X) = P_r(\mathrm{X} \leq \mathrm{x})$, $\mathrm{X} \in [0,1]$. By using Theorem 6.3.1 , we get

$$\left| P_r\left(X \leq \left(\frac{a+b}{2}\right)\right) - (1-\mu) - \left(X - \frac{1}{2}\right) \right| \leq \left(\frac{c(P)}{2(p+1)}\right)^{\frac{1}{P}} \left[\left(\frac{2}{\pi}\right)^{\frac{1}{2p}} \left(e^{\frac{\mu}{\delta}}\right)^{\frac{1}{p}} + 1\right].$$

This implies,

$$\left| P_r\left(X \leq \left(\frac{a+b}{2}\right)\right) + \mu - \mathrm{X} - \frac{1}{2} \right| \leq \left(\frac{c(p)}{2(p+1)}\right)^{\frac{1}{p}} \left[\left(\frac{2}{\pi}\right)^{\frac{1}{2\mathrm{p}}} \left(\mathrm{e}^{\frac{\mu}{\delta}}\right)^{\frac{1}{\mathrm{p}}} + 1\right] \qquad \blacksquare$$

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